

ORBIFOLD GENERA, PRODUCT FORMULAS AND POWER OPERATIONS

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ABSTRACT. We generalize the definition of orbifold elliptic genus, and introduce orbifold genera of chromatic level h , using h -tuples rather than pairs of commuting elements. We show that our genera are in fact orbifold invariants, and we prove integrality results for them. If the genus arises from an H_∞ -map into the Morava-Lubin-Tate theory E_h , then we give a formula expressing the orbifold genus of the symmetric powers of a stably almost complex manifold M in terms of the genus of M itself. Our formula is the p -typical analogue of the Dijkgraaf-Moore-Verlinde-Verlinde formula for the orbifold elliptic genus [DMVV97]. It depends only on h and not on the genus.

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1. INTRODUCTION

This paper aims to provide a systematic understanding and homotopy theoretic refinement of the theory of orbifold genera and product formulas as they arise in string theory (cf. [DMVV97], [Dij99]).

1.1. Product formulas. The most general and famous of these is probably a formula by Dijkgraaf, Moore, Verlinde and Verlinde expressing the orbifold elliptic genus of the symmetric powers of an almost complex manifold M in terms of the elliptic genus of M itself [DMVV97]:

$$\begin{aligned}
 (1) \quad \sum_{n \geq 0} \phi_{\text{ell,orb}}(M^n \wr \Sigma_n) t^n &= \prod_{\substack{m \geq 1, n \geq 0, \\ l \in \mathbb{Z}}} \left(\frac{1}{1 - t^m q^n y^l} \right)^{c(mn, l)} \\
 &= \exp \left[\sum_{m \geq 1} V_m(\phi_{\text{ell}}(M)) t^m \right].
 \end{aligned}$$

Here

$$\phi_{\text{ell}}(M) = \sum_{\substack{n \geq 0, \\ l \in \mathbb{Z}}} c(n, l) q^n y^l$$

is the two-variable elliptic genus or “equivariant χ_y -genus of the loop space” of M . Its definition can be found in [EOTY89], [Höh91] or [HBJ92]. The orbifold version $\phi_{\text{ell,orb}}$ is

defined in [BL03]. The second equation in (1) is due to Borchers¹, and the V_m are a type of Hecke operators acting on q -expansions of Jacobi forms. Borchers proved this equality in the context of his proof of the Moonshine conjectures, and the right-hand side of (1) is often referred to as a *Borchers lift* of $\phi_{\text{ell}}(M)$.

We shall show that the p -typical analogue of (1) arises from a natural equation of cohomology operations in elliptic cohomology by specializing to the elliptic cohomology of a point. Thus our work adds to the evidence that elliptic cohomology has a role to play in the connection between Moonshine and string theory.

Formula (1) has been studied by algebraic geometers [BL03] as well as algebraic topologists [Tam01], [Tam03].

1.2. The Ando-French definition of orbifold genus. In [AF03], Ando and French explain how to fit the notion of orbifold (elliptic) genus into the framework of equivariant elliptic cohomology. The version of equivariant elliptic cohomology they choose to work with is Borel equivariant Morava E -theory E_2 . We explain a slight generalization of Ando and French's definition of orbifold genus: For homotopy theorists, a genus typically is a natural transformation from a cobordism theory to another cohomology theory, applied to a point. If the target of this natural transformation is a form of elliptic cohomology, for instance E_2 , the genus is called an elliptic genus. Let G be a finite group. For each natural number h there is a Morava E -theory E_h , and an element χ of $E_h BG$ can be viewed as a class function on h -tuples of commuting p -power order elements of G (cf. Section 6.2). We write $\mathcal{N}_*^{u,G}$ for the bordism ring of compact, closed, smooth G -manifolds with a complex structure on their stable normal bundle. We write MU_*^G for the coefficients of the complex cobordism spectrum MU_G . There is a Pontrjagin-Thom map from $\mathcal{N}_*^{u,G}$ to MU_*^G and a completion map from MU_*^G to $MU_*(BG)$. For details the reader is referred to Section 3.

Definition 1.1. Let ϕ be a map of ring spectra from MU into E_h . We define the Borel equivariant version ϕ_G of the genus ϕ as the composite

$$\begin{array}{ccc} \phi_G: \mathcal{N}_*^{u,G} \longrightarrow MU_*^G \longrightarrow MU_*(BG) & \xrightarrow{\phi} & E_{h*}(BG) \\ (M \bowtie G) & \longmapsto & \phi_G(M), \end{array}$$

where the first two maps are the Pontrjagin-Thom map and the completion map. We define the orbifold genus associated to ϕ to be

$$\phi_{\text{orb}}(M \bowtie G) := \frac{1}{|G|} \sum_{\alpha} (\phi_G(M))(\alpha),$$

where the sum runs over all h -tuples of commuting elements of p -power order in G .

Note that instead of MU we could have used any of the classical Thom spectra $M\text{Spin}$, MO , $M\text{Sp}$, $MU\langle n \rangle$, $MO\langle n \rangle$ etc.

We shall prove an integrality result about Definition 1.1 and show that it defines in fact an orbifold invariant. While most of the literature on orbifold genera is highly computational, these proofs work on a conceptual level. They rely on deeply homotopy theoretic properties of

¹More precisely, Borchers' computation in the proof of the product formula for the j -function [Bor92, Lemma 7.1] goes through for Jacobi forms, if the Hecke operators are replaced by the V_n defined in [EZ85, I.4.2 (7)].

the $K(\mathfrak{h})$ -local categories, suggesting that stable homotopy theory provides a good framework for the study of orbifold phenomena.

1.3. Power operations. The left-hand side of the DMVV formula (1) involves an object well known to topologists: the assignment

$$M \longmapsto \sum_n (M^n \mathfrak{S} \Sigma_n) t^n$$

is what is called the *total power operation* in cobordism (of a point). The right-hand side of the DMVV formula is a function in $\Phi_{\text{ell}}(M)$ which takes sums into products. Total power operations also have this property. Thus it is a natural question to ask whether formulas like the DMVV formula simply reflect the fact that a natural transformation ϕ preserves power operations. Such a natural transformation that preserves power operations is called an H_∞ -map. We shall show that any H_∞ -map from a cobordism theory into E_h has a DMVV-type formula for the induced (orbifold) genus.

1.4. The chromatic picture. In the case $h = 1$, the cohomology theory E_1 is p -completed K -theory. The standard example of a genus into (non- p -completed) K -theory is the Todd genus. There is an equivariant version of K -theory; here $K_G(\text{pt})$ is the representation ring $R(G)$, and the character of a representation ρ is the class function

$$\chi(g) = \text{Trace}(\rho(g)).$$

Definition 1.1 can then be formulated without the p -power order part, and it becomes the definition of the *topological Todd genus*² of the orbifold $M//G$:

Definition 1.2.

$$\text{Td}_{\text{top}}(M \mathfrak{S} G) := \frac{1}{|G|} \sum_{g \in G} \text{Trace}(g | \text{Td}_G(M)).$$

In the case $h = 2$ Definition 1.1 is (up to the factor $\frac{1}{|G|}$) the Definition [AF03, 6.1]. Ando and French show that this definition is a p -typical analogue of the definition of orbifold elliptic genus discussed in the literature [BL03], [DMVV97]. Thus our point of view fits the *orbifold elliptic genus* (as defined by Ando and French) and its product formula into a common picture with the *topological Todd genus* of an orbifold and its product formula, i.e., the former is exactly the chromatic level two analogue of the latter.

1.5. Statement of results. A priori Td_{top} appears to take values in \mathbb{C} . Note however that

$$\frac{1}{|G|} \sum_{g \in G} \text{Trace}(g | -)$$

equals the inner product with the trivial representation. This shows that Td_{top} takes integral values. In a similar way, using an inner product defined by Strickland, we will prove the following proposition (cf. Corollary 7.11):

Proposition 1.3. *The orbifold genus ϕ_{orb} takes values in E_h^0 .*

²In the literature this turns up as the Euler characteristic of the complex space M/G [AS68a] or the topological Euler characteristic of the orbifold $M//G$ [Dij99].

Definition 1.1 is formulated in terms of the G -space $M \bowtie G$ rather than the orbifold $M//G$. It is a non-trivial fact that ϕ_{orb} depends only on the orbifold, and not on its presentation (cf. Theorem 8.1):

Theorem 1.4. *Let M be a compact complex manifold acted upon by a finite group G , let N be a compact complex manifold acted upon by a finite group H , and assume that the orbifold quotients $M//G$ and $N//H$ are isomorphic as (tangentially almost) complex orbifolds. Then*

$$\phi_{\text{orb}}(M \bowtie G) = \phi_{\text{orb}}(N \bowtie H).$$

The analogues of the Dijkgraaf-Moore-Verlinde-Verlinde formula for our orbifold genera are given by the following theorem (cf. Theorem 9.2):

Theorem 1.5. *For any H_∞ -map ϕ from a Thom spectrum into E_h there is a formula*

$$\sum_{n \geq 0} \phi_{\text{orb}}(M//\Sigma_n) t^n = \exp \left[\sum_{k \geq 0} T_{p^k}(\phi(M)) t^{p^k} \right].$$

There are two side results that are hopefully of independent interest to homotopy theorists. We obtain an explicit formula for the Strickland inner product in Morava E -theory (cf. Corollary 7.13):

Proposition 1.6. *The Strickland inner product in Morava E -theory theory is*

$$\begin{aligned} E_h^0(BG) \otimes E_h^0(BG) &\longrightarrow E_h^0 \\ \chi \otimes \xi &\longmapsto \frac{1}{|G|} \sum_{\alpha} \chi(\alpha) \xi(\alpha), \end{aligned}$$

where the sum is over all h -tuples of commuting elements of p -power order.

The main step in proving that ϕ_{orb} is an orbifold invariant is a theorem about (equivariant) Spanier-Whitehead duals and the Borel construction (cf. Theorem 8.5):

Theorem 1.7. *Let G be a finite group. Then there is an isomorphism of functors from the category of finite G -CW-spectra to the $K(h)$ -local category*

$$EG_+ \wedge_G D_G(-) \cong D(EG_+ \wedge_G(-)),$$

where D_G denotes the G -equivariant dual and D denotes the dual in the $K(h)$ -local category.

In [And92], Ando defines Hecke operators on the Morava E -theories, generalizing the (stable) Adams operations in $K_p^*(X)$. It was pointed out by Atiyah and Tall [AT69] that the Adams operations in K -theory can be defined using the more general theory of λ -rings due to Grothendieck [Gro57]. In Section 9.3, we offer an Atiyah-Tall-Grothendieck type definition for the T_{p^k} , which generalizes Ando's definition to any $K(h)$ -local H_∞ -spectrum.

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Part 1. The topological Todd genus and its product formula

Throughout this paper, the discussion of the (topological) Todd genus and K-theory will serve as a model for our study of elliptic genera and elliptic cohomology.

2. THE TODD GENUS FROM THE POINT OF VIEW OF STABLE HOMOTOPY THEORY

In this section, we recall Conner and Floyd's definition of the Todd genus³ [CF66, I].

2.1. The Conner-Floyd map. In [ABS64], Atiyah, Bott and Shapiro construct K-theory Thom classes u_{ABS} for complex vector bundles. Conner and Floyd [CF66, p.29] show that giving K-theory Thom classes for complex vector bundles is equivalent to giving a map of spectra

$$\text{Td}: \text{MU} \longrightarrow \text{K}$$

(denoted μ_c by Conner and Floyd). On $\text{MU}(\mathfrak{n})$, this map is given by

$$u_{\text{ABS}}(\gamma_{\text{univ}}^{\mathfrak{n}}) \in [\text{MU}(\mathfrak{n}), \mathbb{Z} \times \text{BU}].$$

The map Td is called the Conner-Floyd map. The induced map on homotopy groups,

$$\text{Td}_* := \pi_*(\text{Td}),$$

is the Todd genus.

2.2. The push-forward of one. Assume we are given a multiplicative cohomology theory $E^*(-)$ with natural Thom classes for complex vector bundles, or equivalently, a map of ring spectra $\phi: \text{MU} \rightarrow E$. Let $[\mathbf{M}] \in \text{MU}_{\mathbf{d}}$, i.e., let \mathbf{M} be a compact closed smooth \mathbf{d} -dimensional manifold together with a choice of lift $-\tau]_{\mathbf{K}} \in \widetilde{\mathbf{K}}(\mathbf{M})$ of its stable normal bundle $-\tau] \in \widetilde{\mathbf{KO}}(\mathbf{M})$. Such manifolds are called “manifolds with stably almost complex structure”. Let

$$\pi: \mathbf{M} \longrightarrow \text{pt}$$

denote the unique map from \mathbf{M} to a point. The following is a slight reformulation of the definition of “Umkehr” map along π in [Dye69, pp.40-41], using the language of Thom spaces of virtual bundles set up e.g. in [Rud98, IV].

Definition 2.1. The push-forward along π in $E^*(-)$ is defined by

$$\pi_!^{\phi}: E^*(\mathbf{M}) \xrightarrow{\cong} \widetilde{E}^{*-d}(\mathbf{M}^{-\tau}) \longrightarrow \widetilde{E}^{*-d}(S^0) \cong \pi_{*+d}(E),$$

where the first map is the Thom isomorphism for $-\tau]_{\mathbf{K}}$ and the second map is the Pontrjagin-Thom collapse.

³Conner and Floyd attribute many of the results mentioned here to Atiyah and Hirzebruch [AH61] and Dold [Aar62].

Proposition 2.2. *The genus induced by ϕ ,*

$$\phi_*: \text{MU}_* \longrightarrow \text{E}_*$$

sends $[M] \in \text{MU}_d$ to the push-forward of one $\pi_!^\phi(1) \in \text{E}_d$.

PROOF : The transformation ϕ maps Thom classes to Thom classes and thus $\pi_!^{\text{id}_{\text{MU}}}$ to $\pi_!^\phi$. Therefore it is sufficient to consider the universal case $\phi = \text{id}_{\text{MU}}$. In this case the statement follows directly from the definition of the cobordism Thom classes and from the Pontrjagin-Thom construction. \square

2.3. The classical definition. The Riemann-Roch theorem [Aar62, Dyer] yields the following formula for the Todd genus⁴:

$$\text{Td}(M) = \pi_!^{\text{Td}}(1) = \int_M \prod_i \frac{1 - e^{x_i}}{x_i}.$$

Here the x_i are the Chern roots of the normal bundle ν_M . Let M be a compact complex manifold. Then the index theorem implies:

$$\text{Td}(M) = \sum (-1)^i \dim(H^i(M, \mathcal{O}_M)),$$

where \mathcal{O}_M is the structure sheaf of M (cf. [HBJ92, 5.4] and [AS68a, p.542]).

3. THE EQUIVARIANT TODD GENUS

All the constructions of the previous section go through equivariantly.

3.1. Okonek's equivariant Conner-Floyd maps. The following proposition and examples are taken from [Oko82, 1]⁵.

Proposition 3.1 (Okonek). *If E_G^* is a multiplicative, G -equivariant cohomology theory with natural Thom classes for complex G -bundles, then there is a unique natural, stable transformation*

$$\phi_G: \text{MU}_G^*(-) \longrightarrow \text{E}_G^*(-)$$

of multiplicative G -equivariant cohomology theories that takes Thom classes to Thom classes.

Rather than explaining all the concepts in the statement of the proposition, we state the two examples that are relevant to us.

Example 3.2. For any complex oriented ring spectrum E , Borel equivariant E -cohomology

$$E(\text{EG} \times_G -)$$

has natural Thom classes for complex G -bundles. In this case, ϕ_G factors through Borel equivariant cobordism. If we let ϕ be the orientation of E , then ϕ preserves equivariant Thom classes, so that we get

$$\phi_G: \text{MU}_G(-) \longrightarrow \text{MU}(\text{EG} \times_G -) \xrightarrow{\phi} E(\text{EG} \times_G -).$$

⁴This expression is the inverse of the one Conner and Floyd obtain, because they work with the tangent bundle rather than the normal bundle.

⁵For an English reference see [May96]. There is a difference between the two: Okonek works with tom Dieck's definition of an equivariant cohomology theory [tD71]. In the language of [May96] this is a complex-stable, naive G -equivariant cohomology theory.

Example 3.3. (Compactly supported) equivariant K-theory has natural Thom classes for complex G-bundles. We denote the resulting equivariant Conner-Floyd maps by

$$\mathrm{Td}_G: \mathrm{MU}_G \longrightarrow \mathrm{K}_G.$$

There is a Pontrjagin-Thom map from the equivariant cobordism ring $\mathcal{N}_{*}^{\mathrm{u},G}$ to the coefficient ring MU_G^* , which in the equivariant case fails to be an isomorphism.

Definition 3.4. The equivariant Todd genus of an almost complex G-manifold is defined to be

$$\mathrm{Td}_G(\mathbf{M}) := \mathrm{Td}_{G*}([\mathbf{M}]),$$

where $[\mathbf{M}]$ denotes the image of \mathbf{M} under the G-equivariant Pontrjagin-Thom map.

3.2. Equivariant push-forward of one. The Thom spectrum \mathbf{M}^ξ of a virtual equivariant bundle $\xi \in \mathrm{KO}_G(\mathbf{M})$ and the Thom isomorphism for a choice of stably almost complex structure $[\xi]_K \in \widetilde{\mathrm{K}}_G(\mathbf{M})$ on ξ are defined in [LMSM86, X], and Definition 2.1 goes through for an equivariant theory with Thom classes. On the image of the Pontrjagin-Thom map the same argument as in Proposition 2.2 shows that

$$\phi_G(\mathbf{M}) = \pi_!^{\phi_G}(1) \in \mathrm{E}_G(\mathrm{pt}),$$

where $\pi: \mathbf{M} \rightarrow \mathrm{pt}$ is the unique G-map.

In the case of equivariant K-theory, our definition of push-forward is equivalent to that of Atiyah and Singer⁶ in [AS68b]. Recall that the correct generalization of the Borel construction to G-spectra is given by the “twisted half smash product” over G

$$\mathrm{EG} \ltimes_G -.$$

These twisted half smash products were introduced and studied extensively in [LMSM86]. A summary of their basic properties can be found in [BMMS86, I.1]. For the suspension spectrum of a pointed G-space X, they specialize to the Borel construction

$$\mathrm{EG} \ltimes_G (\Sigma^\infty X) \cong \Sigma^\infty (\mathrm{EG}_+ \wedge_G X).$$

In the case of the Thom spectrum $\mathbf{M}^{-\tau}$ we have (cf. [LMSM86, X.6.3])

$$\mathrm{EG} \ltimes_G (\mathbf{M}^{-\tau}) = (\mathrm{EG} \times_G \mathbf{M})^{-\mathrm{EG} \times_G \tau}.$$

3.3. An explicit formula for Td_G . The equivariant Todd genus takes values in the representation ring $\mathrm{K}_G(\mathrm{pt}) = \mathrm{R}(G)$, and it is classical that a representation $V \rtimes G$ is determined by its character

$$g \longmapsto \mathrm{Trace}(g|V).$$

Using the Riemann-Roch theorem and a Lefschetz fixed point formula, Atiyah, Segal and Singer (cf. [AS68a, (2.11)] and [AS68c]) prove the following:

$$(2) \quad \mathrm{Trace}(g|\mathrm{Td}_G(\mathbf{M})) = \int_{\mathbf{M}^g} \frac{1}{\prod_{\zeta} 1 - \zeta e^{x_j(\mathbf{N}_{\zeta}^g)}},$$

where \mathbf{M}^g stands for the g-fixed points of \mathbf{M} , and \mathbf{N}^g denotes the normal bundle of \mathbf{M}^g in \mathbf{M} ; the product runs over all eigenvalues of the action of g on \mathbf{N}^g and over the Chern roots $x_j(\mathbf{N}_{\zeta}^g)$ of the eigenbundles \mathbf{N}_{ζ}^g .

⁶More precisely, if one replaces TX by X and assumes that all the Thom classes that are needed exist, Atiyah and Singer’s ind_G^X becomes our $\pi_!^X$.

At this point we would like to point out how Definition 1.1 relates to that of Borisov and Libgober [BL03], which looks like the right-hand side of (2). Recall that Definition 1.1 follows the one given by Ando and French, who generalize the left-hand side of (2). Character theory is available in the context of Ando and French's work, but the Riemann-Roch formula is not. However, they explain in detail how to modify the character theoretic discussion in [AS68a] to bring their definition into a form that is modulo a Riemann-Roch theorem very similar to Borisov and Libgober's. Their discussion goes through without changes for our Definition 1.1.

4. POWER OPERATIONS

4.1. Power operations and H_∞ -ring spectra. Let $\{E_G \mid G \text{ finite}\}$ be a compatible family of equivariant cohomology theories in the sense of [LMSM86, II.8.5], and write $E_G(X)$ for $E_G^0(X)$. “Compatible” implies in particular that for a map $\alpha: H \rightarrow G$ and a G -space X , we have a restriction map

$$\text{res}|_\alpha: E_G(X) \longrightarrow E_H(X),$$

and if α is the inclusion of a subgroup and X a G -space we also have an induction map

$$\text{ind}|_H^G: E_H(X) \longrightarrow E_G(X),$$

such that the axioms of a Mackey structure on E_G spelled out in [tD73] are satisfied. The author could not find the reference for this fact, so here is a short explanation: A compatible family satisfies

$$(3) \quad E_G(G \ltimes_H X) \cong E_H(X)$$

for $H \subseteq G$ and any (pointed) H -space X . If X is already a G -space, one has

$$G \ltimes_H X \cong G/H_+ \wedge X,$$

and the map

$$(p_{G/H})_+: G/H_+ \longrightarrow S^0$$

sending all of G/H to the non-basepoint induces $\text{res}|_H^G$, while its G -equivariant Spanier-Whitehead dual

$$D_G(p_{G/H})_+: S^0 \longrightarrow G/H_+$$

induces $\text{ind}|_H^G$. For arbitrary α , the compatibility condition does not provide us with an isomorphism (3), but with a map from the left to the right. Thus if we replace $(p_{G/H})_+$ by the co-unit of the adjunction $(G \ltimes_\alpha -, \text{forget})$, we can still define $\text{res}|_\alpha$. The Mackey criteria follow from [May96, XIX.3]. We also ask that our family has unitary, commutative and associative *external products*

$$\wedge: E_G(X) \otimes E_H(Y) \longrightarrow E_{G \times H}(X \wedge Y),$$

that are natural in (stable) maps of X and Y . Note that this implies that \wedge also commutes with induction and restriction maps. By *unitary* we mean that for each G , there is an element $1 \in E_G(S^0)$ with $1 \wedge x = \text{res}|_{\text{pr}_2} x$, where pr_2 is the projection onto the second factor of $G \times H$. We further ask that $\text{res}|_H^G 1 = 1$.

Example 4.1. For any E , Borel equivariant E -cohomology $E(EG \times_G -)$ is an example [May96, XXI.1.9]. Here the induction maps equal the transfer maps

$$T_H^G: \Sigma_+^\infty(EG \times_G X) \longrightarrow \Sigma_+^\infty(EH \times_H X).$$

Example 4.2. Equivariant K-theory is an example, with the induction maps the induced representation.

Before we recall the definition of an H_∞ -structure on $\{E_G\}$, we need to introduce some notation. Let X be a pointed G -space. We write

$$(X \wr G)^n \wr \Sigma_n \quad \text{or} \quad X^n \wr (G \wr \Sigma_n)$$

for the space $X^{\wedge n}$ acted on by

$$G \wr \Sigma_n = G^n \rtimes \Sigma_n$$

as follows: G acts on each factor individually, while Σ_n permutes the factors. By abuse of notation, we also write $E_{\Sigma_n}(X^n)$ for $E_{G \wr \Sigma_n}(X^n)$ and in particular $E(X)$ for $E_G(X)$, unless we want to emphasize the equivariant situation. The following definition is essentially [BMMS86, VIII.1.1].

Definition 4.3. An H_∞ -structure on E is given by a collection of natural maps

$$P_n: E_G(X) \longrightarrow E_{G \wr \Sigma_n}(X^n)$$

called *power operations* satisfying the following conditions:

- (a) $P_1 = \text{id}$ and $P_0(x) = 1$,
- (b) the (external) product of two power operations is

$$P_j(x) \wedge P_k(x) = \text{res}_{\Sigma_j \times \Sigma_k}^{\Sigma_{j+k}} (P_{j+k}(x)),$$

- (c) the composition of two power operations is

$$P_j(P_k(x)) = \text{res}_{\Sigma_k \wr \Sigma_j}^{\Sigma_{jk}} (P_{jk}(x)),$$

- (d) and the P_j 's preserve (external) products:

$$P_j(x \wedge y) = \text{res}_{\Sigma_j}^{\Sigma_j \times \Sigma_j} (P_j(x) \wedge P_j(y)),$$

where the restriction is along the map

$$[((X \wr G)^2)^j \wr \Sigma_j] \longrightarrow [(X \wr G)^{2j} \wr (\Sigma_j \times \Sigma_j)] \cong [((X \wr G)^j \wr \Sigma_j)^2].$$

Remark 4.4. Traditionally people formulated this definition only for Borel equivariant theories. In that case it is a refinement of the notion of ring spectrum up to homotopy, but it is weaker than the notion of E_∞ or A_∞ structure. In the same way our definition is weaker than Greenlees and May's notion of *global \mathcal{I}_* functor with smash product spectrum* in [GM97]. More precisely, using the Yoneda lemma one can reformulate Definition 4.3 in terms of maps of $G \wr \Sigma_n$ -spectra

$$\xi_n: E_G^n \longrightarrow E_{G \wr \Sigma_n}.$$

The maps $\text{res}|_H^G$ are induced by maps of G -spectra

$$G \rtimes_H E_H \longrightarrow E_G,$$

and the external product \wedge becomes a map of $G \times H$ -spectra. Thus conditions (1)-(4) of the definition translate into homotopy commutative diagrams of spectra. A global \mathcal{I}_* functor with smash product spectrum has such ξ_n , and in that case, the diagrams commute strictly.

4.2. Total power operations. Let E be an H_∞ -ring spectrum. It is often convenient to consider all power operations at once, i.e. the *total power operation*

$$P: E(X) \longrightarrow \bigoplus_{n \geq 0} E_{\Sigma_n}(X^n) t^n$$

which is P_n into each summand. Here we are following the notation of [Seg96]: The symbol \bigoplus stands for the infinite product, and the variable t is a dummy variable, introduced in order to keep track of the “summand” and also to avoid convergence issues later on. Note that

$$\bigoplus_{n \geq 0} E_{\Sigma_n}(X^n) t^n$$

is a graded ring by

$$E_{\Sigma_n}(X^n) \otimes E_{\Sigma_m}(X^m) \xrightarrow{\wedge} E_{\Sigma_n \times \Sigma_m}(X^{n+m}) \xrightarrow{\text{ind}} E_{\Sigma_{n+m}}(X^{n+m}),$$

where

$$\text{ind} = \text{ind} |_{\Sigma_n \times \Sigma_m}^{\Sigma_{n+m}}$$

(compare [Seg96]).

Proposition 4.5 (compare [BMMS86, VIII.1.1]). *We have*

(a) *the restriction of $P_j(x)$ to $E(X^j)$ is*

$$\text{res} |_1^{\Sigma_j} P_j(x) = x^{\wedge j},$$

(b) *the operation P_j applied to $1 \in E^0(S^0)$ is*

$$P_j(1) = 1_{\Sigma_j} := 1 \in E_{\Sigma_j}(\text{pt}),$$

(c) *the total power operation takes sums into products:*

$$P(x + y) = P(x) \cdot P(y).$$

PROOF : The first two properties are immediate from the definition. The proof of property (c) in [BMMS86, II.1.6] and [LMSM86, VII.1.10] takes place on the level of equivariant spectra. \square

4.3. Power operations in K-theory and cobordism. In [Ati66] Atiyah defines power operations for K-theory. In the case of an (equivariant) vector bundle V over (a G -space) X , they are given by the (external) tensor product

$$P_n([V]) = [V^{\otimes n}] \in K_{\Sigma_n}(X^n).$$

In [tD68] tom Dieck defines power operations for Borel equivariant cobordism and shows that the Conner-Floyd map is an H_∞ -map in the classical (i.e. Borel equivariant) sense. We prefer to work on the level of equivariant cobordism $MU_G^*(-)$ (cf. [tD70]). In that case, Greenlees and May show that MU_G is a “global \mathcal{I}_* functor with smash product spectrum”

[GM97, 5.8]. Thus, by Remark 4.4 it has power operations. On coefficients⁷ these power operations in equivariant cobordism are

$$\begin{aligned} P_n: MU^*(pt) &\longrightarrow MU_{\Sigma_n}^{n*}(pt) \\ [M] &\longmapsto [M^n \wr \Sigma_n]. \end{aligned}$$

Proposition 4.6 (compare [tD68, (A4)]). *The P_n are multiplicative with respect to \wedge and compatible with Thom classes in the following sense:*

$$(4) \quad P_n(u_{MU}(\xi)) = u_{MU, \Sigma_n}(\xi^{\oplus n}),$$

and $\{MU_G\}$ equipped with the P_n is universal with respect to this property. In other words, for any equivariant cohomology theory with multiplicative Thom classes for complex G -bundles and power operations satisfying (4) the maps ϕ_G of Proposition 3.1 preserve power operations.

PROOF : If ξ is the universal complex G -bundle, Equation (4) is immediate from the construction; for other ξ it follows by naturality. Let now $\{E_G\}$ be an equivariant family as in the proposition. Then the proof of Proposition 3.1 shows that the ϕ_G preserve power operations. \square

Corollary 4.7 (compare [tD68]). *The map*

$$MU_G^*(-) \longrightarrow MU^*(EG \times_G -)$$

of Example 3.2 and the equivariant Conner-Floyd-Okonek map

$$Td_G: MU_G \longrightarrow K_G$$

of Example 3.3 are H_∞ -maps.

PROOF : Both maps are defined as examples of the map ϕ_G , which is an H_∞ -map by the proof of Proposition 4.6. \square

Let E be an H_∞ -spectrum with compatible Thom classes as in Proposition 4.6, and let V be a complex d -dimensional G -representation. Then E_G comes equipped with natural isomorphisms

$$E_G^0(\mathbb{S}^{2d} \wedge X) \xrightarrow{\cong} E^0(V^c \wedge X),$$

where V^c denotes the one point compactification of V (cf. [GM97, 2.1]). This becomes important when we want to extend our power operations to

$$E^{-2d}(X) = E^0(\mathbb{S}^{2d} \wedge X),$$

because $(\mathbb{S}^{2d})^n \wr \Sigma_n$ is an equivariant sphere. In the situation of the proposition we can follow [GM97] to extend the power operations to

$$P_n: E^{2d}(X) \longrightarrow E^{2nd}(X^n).$$

⁷More precisely: on non-equivariant coefficients or on the image of the Pontrjagin-Thom map.

4.4. Internal power operations. We can always compose the power operation P_n with the pullback along the diagonal map of X^n

$$\Delta_n^*: E_{\Sigma_n}(X^n) \longrightarrow E_{\Sigma_n}(X).$$

Since the action of Σ_n on X is trivial, the target of this map often turns out to be

$$E_{\Sigma_n}(\text{pt}) \otimes E(X).$$

We might want to compose further with a map

$$E_{\Sigma_n}(\text{pt}) \longrightarrow E^0$$

in order to obtain operations⁸ acting on $E(X)$. In the case of K -theory, $E_{\Sigma_n}(\text{pt})$ is the representation ring $R(\Sigma_n)$. The following two examples from [Ati66] are important to us.

Example 4.8. Atiyah's definition of the Adams operations is

$$\psi_n(x) = \text{Trace}(c_n | \Delta_n^* P_n(x)),$$

where c_n is a cycle of length n .

Example 4.9. The operations σ_n are defined by

$$\begin{aligned} \sigma_n(x) &:= \frac{1}{n!} \sum_{g \in \Sigma_n} \text{Trace}(g | \Delta_n^* P_n(x)) \\ &= \langle \Delta_n^* P_n(x), 1 \rangle_{\Sigma_n}. \end{aligned}$$

If $x = [V]$ is the class of a vector bundle V , then

$$\sigma_n(x) = [\text{Sym}^n(V)]$$

is represented by the n^{th} symmetric power of V , since in this case the inner product with 1 counts the multiplicity of the trivial representation as a summand of

$$[V^{\otimes n} \rtimes \Sigma_n] = \Delta_n^* P_n(x) \in R(\Sigma_n) \otimes K(X).$$

Definition 4.10. We write

$$S_t(x) := \sum_{n \geq 0} \sigma_n(x) t^n$$

for the total symmetric power. In other words, S_t is the composite

$$S_t: K(X) \xrightarrow{P} \bigoplus_{n \geq 0} K_{\Sigma_n}(X^n) t^n \longrightarrow \bigoplus_{n \geq 0} R(\Sigma_n) \otimes K(X) t^n \longrightarrow K(X)[[t]],$$

where on the n^{th} summand the second map is pullback along the diagonal and the third map is the inner product with 1_{Σ_n} .

In [And92], Ando generalizes Atiyah's work to cohomology theories with Hopkins-Kuhn-Ravenel character theory, as we will recall in Section 6.5.

⁸In the literature (e.g. [And92]) these compositions are often referred to as power operations and P_n is then called "total power operation". We follow the convention to call them *internal power operations*, since they actually act on $E(X)$.

5. MULTIPLICATIVE FORMULAS FOR THE TODD GENUS

The following is a reformulation of the second statement of Corollary 4.7:

Corollary 5.1. *The square*

$$\begin{array}{ccc}
 \mathrm{MU}(\mathbf{X}) & \xrightarrow{\mathrm{Td}} & \mathrm{K}(\mathbf{X}) \\
 \mathrm{P}_{\mathrm{MU}} \downarrow & & \downarrow \mathrm{P}_{\mathrm{K}} \\
 \bigoplus_{n \geq 0} \mathrm{MU}_{\Sigma_n}(\mathbf{X}^n) \mathbf{t}^n & \xrightarrow{\bigoplus_{n \geq 0} \mathrm{Td}_{\Sigma_n}} & \bigoplus_{n \geq 0} \mathrm{K}_{\Sigma_n}(\mathbf{X}^n) \mathbf{t}^n
 \end{array}$$

commutes.

It follows immediately that the equivariant Todd genera $\mathrm{Td}_{\Sigma_n}(\mathbf{M}^n)$ are determined by the Todd genus of \mathbf{M} , and moreover that the expression is exponential in $\mathrm{Td}(\mathbf{M})$. More precisely, specializing to the case where \mathbf{X} is a point results in the following corollary:

Corollary 5.2. *Let \mathbf{M} be an almost complex manifold. Then we have the following equation in the ring $\bigoplus_{n \geq 0} \mathbf{R}(\Sigma_n) \mathbf{t}^n$:*

$$\sum_{n \geq 0} \mathrm{Td}_{\Sigma_n}(\mathbf{M}^n) \mathbf{t}^n = \left(\sum_{n \geq 0} \mathbf{1}_{\Sigma_n} \mathbf{t}^n \right)^{\mathrm{Td}(\mathbf{M})},$$

where $\mathbf{1}_{\Sigma_n} \in \mathbf{R}(\Sigma_n)$ denotes the trivial representation.

PROOF : By Corollary 5.1 we have

$$\sum_{n \geq 0} \mathrm{Td}_{\Sigma_n}(\mathbf{M}^n) \mathbf{t}^n = \mathrm{P}_{\mathrm{K}}(\mathrm{Td}(\mathbf{M})).$$

By Proposition 4.5 (c), P_{K} takes sums into products. Since

$$\mathrm{Td}(\mathbf{M}) \in \mathrm{K}(\mathrm{pt}) = \mathbb{Z},$$

this implies

$$\mathrm{P}_{\mathrm{K}}(\mathrm{Td}(\mathbf{M})) = \mathrm{P}_{\mathrm{K}}(\mathbf{1})^{\mathrm{Td}(\mathbf{M})}.$$

Now $\mathrm{P}_{\mathrm{K}}(\mathbf{1})$ is the trivial representation of Σ_n (compare Proposition 4.5 (b)). Therefore,

$$\mathrm{P}_{\mathrm{K}}(\mathbf{1}) = \sum_{n \geq 0} \mathbf{1}_{\Sigma_n} \mathbf{t}^n.$$

□

As a further consequence of Corollary 5.1, we obtain the multiplicative formula for the topological Todd genus [Dij99]:

Corollary 5.3. *We have*

$$\sum_{n \geq 0} \mathrm{Td}_{\mathrm{top}}(\mathbf{M}^n \bowtie \Sigma_n) \mathbf{t}^n = \left(\frac{1}{1 - \mathbf{t}} \right)^{\mathrm{Td}(\mathbf{M})} = \exp \left[\sum_{n \geq 1} \frac{\psi_n(\mathrm{Td}(\mathbf{M}))}{n} \mathbf{t}^n \right].$$

This is the chromatic level one analogue of the DMVV formula (1).

PROOF : We have

$$\begin{aligned}
(5) \quad \sum_{n \geq 0} \mathrm{Td}_{\mathrm{top}}(M^n \mathfrak{S} \Sigma_n) t^n &= \sum_{n \geq 0} \frac{1}{n!} \sum_{g \in \Sigma_n} \mathrm{Trace}(g | \mathrm{Td}_{\Sigma_n}(M^n)) t^n \\
&= \sum_{n \geq 0} \frac{1}{n!} \sum_{g \in \Sigma_n} \mathrm{Trace}(g | P_n(\mathrm{Td}(M))) t^n \\
&= S_t(\mathrm{Td}(M)),
\end{aligned}$$

where the first equation is the definition of $\mathrm{Td}_{\mathrm{top}}$, the second equation is Corollary 5.1, and the third equation is Definition 4.10 with X the one point space.

The first identity of the corollary now follows exactly like Corollary 5.2 from the fact that S_t is exponential. We thank Charles Rezk for reminding us of the well-known equation

$$S_t(x) = \exp \left[\sum_{n \geq 1} \frac{\psi_n(x)}{n} t^n \right].$$

Together with (5) this proves

$$\sum_{n \geq 0} \mathrm{Td}_{\mathrm{top}}(M^n \mathfrak{S} \Sigma_n) t^n = \exp \left[\sum_{n \geq 1} \frac{\psi_n(\mathrm{Td}(M))}{n} t^n \right].$$

□

Part 2. The orbifold elliptic genus and other higher chromatic relatives of $\mathrm{Td}_{\mathrm{top}}$

The methods of Part 1 appear to be specific to equivariant K -theory: We use the inner product of two representations, symmetric powers of vector bundles, and evaluation of characters at group elements. Our discussion in the higher chromatic case relies on the fact that character theory as well as inner products have been defined in much greater generality. Firstly, for E a suitable $K(\mathfrak{h})$ -local cohomology theory, e.g. Morava E -theory $E_{\mathfrak{h}}$, and χ an element of $E^0(BG)$, Hopkins-Kuhn-Ravenel theory defines evaluation of χ at \mathfrak{h} -tuples of commuting p -power order elements of G (cf. Section 6.2). Secondly, Strickland has defined inner products

$$b_G: E^0(BG) \otimes E^0(BG) \longrightarrow E^0$$

in any $K(\mathfrak{h})$ -local cohomology theory E (cf. Section 7.2). If Hopkins-Kuhn-Ravenel theory applies and E^0 is torsion free they satisfy the formula

$$(6) \quad b_G(\chi, \xi) = \frac{1}{|G|} \sum_{\alpha} \chi(\alpha) \xi(\alpha),$$

where the sum runs over all \mathfrak{h} -tuples of commuting p -power order elements of G (cf. Corollary 7.13).

In Sections 6.5 and 7, we recall how to use Hopkins-Kuhn-Ravenel character theory to define orbifold genera ϕ_{orb} and Hecke operators in Morava E -theory. We also generalize the definition of symmetric powers to operations in Morava E -theory. If ϕ is an H_{∞} -map we prove a DMVV-type product formula for ϕ_{orb} (cf. Section 9.2). The formula (6) implies

integrality statements for ϕ_{orb} and the symmetric powers (cf. Corollaries 7.11 and 7.12). Another consequence of (6) is the key fact that the map

$$\chi \mapsto \frac{1}{|\mathbf{G}|} \sum_{\alpha} \chi(\alpha)$$

is induced by a map in the $\mathbf{K}(\mathbf{h})$ -local category. It will play a central role in Section 8, where we prove that $\phi_{\text{orb}}(\mathbf{M}^n \mathfrak{S} \Sigma_n)$ does not depend on the representation of the orbifold $\mathbf{M} // \mathbf{G}$. It will also allow us to generalize the definitions of symmetric powers and Hecke operators to any $\mathbf{K}(\mathbf{h})$ -local \mathbf{H}_{∞} -spectrum \mathbf{E} (cf. Sections 7.5 and 9.3).

6. HOPKINS-KUHN-RAVENEL THEORY

This section recalls some results from [HKR00]. The reader can find a nice and short introduction to Hopkins-Kuhn-Ravenel character theory in [AF03, 5], also see [Rez, 8].

6.1. Even periodic ring spectra and formal groups. We keep our paper in the language of even periodic ring spectra, because all our examples are of this kind. This section is a short reminder of their definition and properties. For details see [AHS01].

Definition 6.1. An even periodic ring spectrum is a spectrum \mathbf{E} such that the graded coefficient ring \mathbf{E}^* is concentrated in even degrees and \mathbf{E}^2 contains a unit.

No choice of this unit is specified. In the context of even periodic ring spectra it is often convenient to replace the complex cobordism spectrum \mathbf{MU} by its two-periodic version

$$\mathbf{MP} := \bigvee_{j \in \mathbb{Z}} \Sigma^{2j} \mathbf{MU}.$$

Note that \mathbf{MP} is the Thom spectrum of $\mathbb{Z} \times \mathbf{BU}$. For even periodic \mathbf{E} the Atiyah-Hirzebruch spectral sequence for $\mathbf{E}^*(\mathbb{CP}^n)$ collapses, and the system

$$\mathbf{E}^*(\mathbb{CP}^n) \longleftarrow \mathbf{E}^*(\mathbb{CP}^{n+1})$$

is Mittag-Leffler, such that $\mathbf{E}^*(\mathbb{CP}^{\infty})$ becomes non-canonically isomorphic to $\mathbf{E}^*[[x]]$. As usual⁹ a good choice of such an x gives rise to \mathbf{E} -theory Chern classes, and to a formal group law F over \mathbf{E}_* describing the first Chern class of the tensor product of line bundles

$$c_1(L_1 \otimes L_2) = c_1(L_1) +_F c_1(L_2).$$

The advantage of working with even periodic \mathbf{E} is that rather than speaking about formal group laws one can use the language of formal groups: For such \mathbf{E} the map

$$\mathbb{CP}^{\infty} \times \mathbb{CP}^{\infty} \longrightarrow \mathbb{CP}^{\infty}$$

classifying the tensor product of line bundles makes the formal spectrum $\text{spf } \mathbf{E}^0(\mathbb{CP}^{\infty})$ into an (affine one-dimensional) formal group scheme, and choosing a coordinate for this formal group is equivalent to specifying a map of ring spectra

$$\mathbf{MP} \longrightarrow \mathbf{E}.$$

We do not make much use of these concepts, but we use several results whose proofs rely on a deep understanding of the way these formal groups come into the picture. For the moment

⁹Cf. [Ada95], [Rud98].

it is enough to remember that an even periodic ring spectrum E has somehow a formal group attached to it.

6.2. Morava E-theories. We now explain which spectra we can work with.

Definition 6.2. Let E be an even periodic ring spectrum with associated formal group F . We say that E has a *Hopkins-Kuhn-Ravenel theory* if

- (a) E^0 is local with maximal ideal \mathfrak{m} , and complete in the \mathfrak{m} -adic topology,
- (b) the graded residue field E^0/\mathfrak{m} has characteristic $p > 0$,
- (c) $p^{-1}E^0$ is not zero,
- (d) the mod \mathfrak{m} reduction of F has height $h < \infty$ over E^0/\mathfrak{m} .

Hopkins, Kuhn and Ravenel give a list of examples satisfying the conditions of this definition. One of these examples is in addition an H_∞ -spectrum and the interplay between Hopkins-Kuhn-Ravenel theory and the H_∞ -structure is well understood. This is the reason why it becomes our favorite example:

Example 6.3 (Lubin-Tate cohomology/Morava E-theory). Consider the graded ring

$$E_* := \mathbb{W}\mathbb{F}_{p^h}[[u_1, \dots, u_{h-1}]]\langle u^{\pm 1} \rangle,$$

where u_i has degree zero, u has degree 2, and $\mathbb{W}k$ denotes the ring of Witt vectors of the field k . There is a cohomology theory called *Lubin-Tate cohomology* or *Morava E-theory*, which has E_* as coefficients. On a finite complex X , it is given by

$$E_h^*(X) = MU^*(X) \otimes_{MU^*} E^*,$$

where the map $MU_* \rightarrow E_*$ classifies the universal deformation of the Honda formal group law. The construction of this cohomology theory goes back many years, a published account can be found in¹⁰ [Rez98].

6.3. h -tuples of commuting elements. Just as classical characters of G are class functions on G , Hopkins-Kuhn-Ravenel characters are class functions on h -tuples of commuting p -power order elements of G . This section is a short reminder of the basic definitions concerning such h -tuples; we will give a more detailed discussion of the case $G = \Sigma_n$ in Section 9.1. Since G is finite, the set of all h -tuples of commuting elements of p -power order of G can be identified with

$$\mathrm{Hom}(\mathbb{Z}_p^h, G).$$

The group G acts on this set by conjugation:

$$g(g_1, \dots, g_h)g^{-1} = (gg_1g^{-1}, \dots, gg_hg^{-1}).$$

Definition 6.4. Let α be an h -tuple of commuting elements (of p -power order) of G . The conjugacy class $[\alpha]_G$ of α is defined to be the orbit of α in $\mathrm{Hom}(\mathbb{Z}^h, G)$ (or $\mathrm{Hom}(\mathbb{Z}_p^h, G)$ respectively) under this G action. The centralizer of α is defined as the stabilizer

$$C_\alpha = C_G(\alpha) := \mathrm{Stab}_G(\alpha) \subseteq G.$$

Definition 6.5. A function on $\mathrm{Hom}(\mathbb{Z}_p^h, G)$ is called a class function if it is invariant under conjugation by elements of G .

¹⁰Rezk omits a subtlety in his exposition: He proves that E_* is Landweber exact over BP, obtaining a homology theory. Via Spanier-Whitehead duality this becomes a cohomology theory on finite complexes as described. The phantom discussion in [HS99] proves that it is (uniquely) represented by a ring spectrum.

6.4. Hopkins-Kuhn-Ravenel characters. Let E be a spectrum with Hopkins-Kuhn-Ravenel theory, let G be a finite group, let χ be an element of $E^0(BG)$ and let α be an h -tuple of commuting p -power order elements of G , where h is as in Definition 6.2. Then Hopkins, Kuhn and Ravenel define a ring D and an evaluation map

$$\begin{aligned} \text{eval}_\alpha: E^0(BG) &\longrightarrow D \\ \chi &\longmapsto \chi(\alpha). \end{aligned}$$

For our purposes it is not important what the ring D is or how eval_α is defined, but for completeness, we recall their definitions: Let D_n be the ring

$$D_n := E^0(B(\mathbb{Z}/p^n\mathbb{Z})^2)/(\text{annihilators of nontrivial Euler classes}),$$

then

$$D = \varinjlim_n D_n$$

is the colimit over the maps induced by

$$\mathbb{Z}/p^{n+1}\mathbb{Z} \longrightarrow \mathbb{Z}/p^n\mathbb{Z}.$$

Since G is finite, any $\alpha \in \text{Hom}(\mathbb{Z}_p^h, G)$ factors through some $\alpha_n \in \text{Hom}((\mathbb{Z}/p^n\mathbb{Z})^h, G)$, and

$$\text{eval}_\alpha(\chi) := \alpha_n^*(\chi) \in D$$

is independent of the choice of n .

We will use the fact that D is independent of the group G and that a fixed $\chi \in E^0(BG)$ defines a class function on the set of h -tuples of commuting p -power order elements of G . This is the sense in which χ is a character. The maps eval_α are analogues of the $\text{Trace}(g| -)$ maps in representation theory. The following is a corollary of [HKR00, Thm C].

Theorem 6.6. *Let E be a ring spectrum with Hopkins-Kuhn-Ravenel theory. An element χ of $\frac{1}{p}E^0(BG)$ is uniquely determined by the class function it defines.*

We also need the Hopkins-Kuhn-Ravenel analogue of the formula for the character of an induced representation [Ser77, p.30].

Theorem 6.7 ([HKR00, Thm D]). *Let $H \subseteq G$ be a subgroup, and let α be an h -tuple of commuting p -power order elements in G . We have*

$$(\text{ind}_H^G(\chi))(\alpha) = \frac{1}{|H|} \sum_{\substack{g \in G | g\alpha g^{-1} \\ \text{maps to } H}} \chi(g\alpha g^{-1}).$$

6.5. Ando's generalization of Atiyah's work. The original reference for this Section is [And92], see also [And95]. Let E be an H_∞ -ring spectrum with Hopkins-Kuhn-Ravenel theory. For simplicity we assume a Künneth isomorphism for the symmetric groups, i.e., we ask that $E^0(B\Sigma_n)$ be free of finite rank over E^0 and that $E^1(B\Sigma_n) = 0$.

Example 6.8. ([Str98, 3.3],[And95]) The Morava E -theories E_n satisfy all the above conditions.

For such spectra, Ando defines internal power operations. The examples relevant to us are the analogues of the examples in Section 4.4.

Definition 6.9. Let α be an \mathbf{h} -tuple of commuting elements of \mathbf{p} -power order of Σ_n . Define

$$\psi_\alpha: E(X) \longrightarrow D \otimes E(X)$$

as the composition

$$E(X) \xrightarrow{p_n} E(E\Sigma_n \times_{\Sigma_n} X^n) \xrightarrow{\Delta_n^*} E(B\Sigma_n \times X) \xleftarrow{\cong} E(B\Sigma_n) \otimes E(X) \longrightarrow D \otimes E(X),$$

where Δ_n denotes the diagonal map of X^n , and the last arrow sends $\chi \otimes x$ to $\chi(\alpha) \otimes x$.

Let α be as above. Then α makes $\{1, \dots, n\}$ into a $\mathbb{Z}_p^{\mathbf{h}}$ -set. Conversely, a finite $\mathbb{Z}_p^{\mathbf{h}}$ -set A determines an \mathbf{h} -tuple of \mathbf{p} -power elements α of some symmetric group up to conjugacy (cf. Section 9.1). We sometimes write ψ_A for ψ_α .

Definition 6.10. The Hecke operators in Morava-Lubin-Tate theory are defined as

$$T_{p^k}(x) := \frac{1}{p^k} \sum_{\substack{T \in \mathcal{T}_p \\ |T|=p^k}} \psi_T(x),$$

where the sum is over all isomorphism classes of transitive $\mathbb{Z}_p^{\mathbf{h}}$ -sets of order p^k . It is proved in [And92] that these T_{p^k} are additive operations

$$T_{p^k}: E_{\mathbf{h}}(X) \longrightarrow E_{\mathbf{h}}(X).$$

Note that on $E_1 = K_p^\wedge$, these Hecke operators are the stable Adams operations:

$$T_{p^k} = \frac{\psi_{p^k}}{p^k}.$$

Definition 6.11. Let E^0 be torsion free. We define the analogues of the symmetric powers as

$$\sigma_n(x) := \frac{1}{n!} \sum_{\alpha} \psi_\alpha(x),$$

where $x \in E(X)$, and this time the sum runs over all \mathbf{h} -tuples α of commuting elements of \mathbf{p} -power order in Σ_n . We write S_t for the “total symmetric power” as above.

It is immediate from [AF03, 5.5] that the operation σ_n takes values in $\frac{1}{|\Sigma_n|} E^0(X)$, but it is a non-trivial fact that it takes values in $E^0(X)$. We postpone the proof to Section 7.3. As in the case of K -theory, S_t turns out to take sums into products. We will give a more general definition of the σ_n and prove this exponential property in Section 7.5.

7. GENERALIZED ORBIFOLD GENERA

Recall from Definition 1.1 that if M is a stably almost complex oriented G -manifold, and

$$\phi: MU \longrightarrow E_{\mathbf{h}}$$

is a complex orientation of Morava E -theory, then the orbifold genus $\phi_{\text{orb}}(M \bowtie G)$ is defined by the formula

$$\phi_{\text{orb}}(M \bowtie G) = \frac{1}{|G|} \sum_{\alpha} (\phi_G(M))(\alpha),$$

where ϕ_G is the Borel equivariant version of ϕ , and the sum runs over all \mathbf{h} -tuples of commuting elements of \mathbf{p} -power order in G . In this section, we generalize the definitions of

ϕ_{orb} and of σ_n , using maps in the $K(\mathfrak{h})$ -local category. As a corollary of these new definitions, we obtain the promised integrality statements.

7.1. The $K(\mathfrak{h})$ -local categories. Let $H_*(-)$ be a generalized homology theory. Recall from [Bou79] that there is a category \mathcal{S}_H , called the H -local (stable homotopy) category, and a functor

$$\gamma: \mathcal{S} \longrightarrow \mathcal{S}_H,$$

which is left-universal with respect to the property that it takes H_* -isomorphisms into isomorphisms. When it is clear that we are working in \mathcal{S}_H , we will often omit γ from the notation. Like the stable homotopy category \mathcal{S} itself, \mathcal{S}_H is a triangulated category with a compatible closed symmetric monoidal structure. In other words, it has a symmetric monoidal structure $-\wedge-$ with unit $S = \gamma(S^0)$ and function objects (“internal hom’s”) $F(-, -)$, such that

$$\text{Hom}(X \wedge Y, Z) = \text{Hom}(X, F(Y, Z)),$$

and these data are compatible with the triangulated structure in an appropriate sense¹¹. The localization functor γ preserves the triangulated structure as well as the monoidal structure and its unit, but does not in general preserve function objects¹². There is one important class of function objects preserved by γ , which is going to play a role for us: Write

$$DX := F(X, S)$$

for the dual of X .

Theorem 7.1 ([LMSM86, III.1.6]). *Let X and Y be objects of a closed symmetric monoidal category, and assume that there are maps*

$$\alpha: S \longrightarrow X \wedge Y \quad \text{and} \quad \beta: Y \wedge X \longrightarrow S$$

such that the composites

$$(\text{id} \wedge \beta) \circ (\alpha \wedge \text{id}): X \cong S \wedge X \longrightarrow X \wedge Y \wedge X \longrightarrow X \wedge S \cong X$$

and

$$(\beta \wedge \text{id}) \circ (\text{id} \wedge \alpha): Y \cong Y \wedge S \longrightarrow Y \wedge X \wedge Y \longrightarrow S \wedge Y \cong Y$$

are the respective identity maps. Then the adjoint $\beta^\sharp: Y \rightarrow DX$ is an isomorphism.

An object X for which such Y , β and α exist is called *strongly dualizable*. It comes with an isomorphism $X \rightarrow DD X$. Since γ preserves the monoidal structure, Theorem 7.1 implies that γ also preserves strong dualizability and strong duals.

Definition 7.2. Let E be a spectrum such that any map that becomes an isomorphism under $H_*(-)$ also becomes an isomorphism under $E^*(-)$. Then E is called an H -local spectrum.

If E is H -local, $E^*(-)$ is a well-defined functor on the category \mathcal{S}_H . The following theorem seems to be well-known to homotopy theorists¹³:

Theorem 7.3. *Let E be a cohomology theory with level \mathfrak{h} Hopkins-Kuhn-Ravenel character theory. Then E is local with respect to the Morava K -theory $K(\mathfrak{h})$.*

¹¹The details can be found in [HPS97, A.2].

¹²Cf. [HPS97, 3.5.1].

¹³To the author’s knowledge there is no published account of it. In the case that E is Morava-Lubin-Tate cohomology it is proved in [HS99, 5.2], for Noetherian E^0 a written account is available from [Str04], in the generality it is stated here I learned it from Michael Hopkins.

These *Morava K-theory* homology theories $K(h)_*(-)$ were first constructed by Baas and Sullivan and first used by Morava. Today their definition can be found in [Rud98] or [EKMM97]. The functor γ has a fully faithful right-adjoint J , whose image is the (full) subcategory of $K(h)$ -local spectra, and it is customary to think of $\mathcal{S}_{K(h)}$ as embedded into \mathcal{S} via J . This point of view is not helpful for our purposes, and we stick to the language of localized categories. The difference is mainly in notation: Write $L_{K(h)}$ for the composite $J \circ \gamma$. The functor J does not preserve the monoidal structure. Thus, where we write

$$\gamma(X) \wedge \gamma(Y) \quad \text{or} \quad X \wedge Y$$

for the smash product in $\mathcal{S}_{K(h)}$, others write

$$L_{K(h)}(L_{K(h)}X \wedge L_{K(h)}Y),$$

and similarly we write \mathbb{S}^0 or $\gamma(\mathbb{S}^0)$ for $L_{K(h)}\mathbb{S}^0$.

7.2. Strickland inner products. This section recalls some of the concepts and results in [Str00]. Let \mathcal{C} be an additive closed symmetric monoidal category. We use the notation of the previous section and write τ for the twist map $X \wedge Y \rightarrow Y \wedge X$. We fix the assumption on \mathcal{C} that every object is strongly dualizable.

Definition 7.4. A *Frobenius object* in \mathcal{C} is an object A equipped with maps

$$S \xrightarrow{\eta} A, \quad A \wedge A \xrightarrow{\mu} A, \quad A \xrightarrow{\varepsilon} S, \quad \text{and} \quad A \xrightarrow{\psi} A \wedge A$$

such that

- (a) (A, η, μ) is a commutative and associative monoid,
- (b) (A, ε, ψ) is a commutative and associative co-monoid,
- (c) we have $\psi \circ \mu = (1 \wedge \mu) \circ (\psi \wedge 1)$.

Lemma 7.5 ([Str00, 3.9]). *If $(A, \eta, \mu, \psi, \varepsilon)$ is a Frobenius object in \mathcal{C} then $b := \varepsilon\mu$ defines an inner product on A in the following sense:*

- (a) b is symmetric, i.e., $b \circ \tau = b$, and
- (b) b is non-degenerate, i.e., the adjoint $b^\sharp: A \rightarrow DA$ is an isomorphism.

Let G be a finite group or groupoid, and let BG denote its Borel construction. We write BG_+ for the $K(h)$ -local suspension spectrum of the Borel construction of G :

$$\gamma(\Sigma_+^\infty BG).$$

Let $\delta: G \rightarrow G \times G$ denote the inclusion of the diagonal, and write ψ for $B\delta_+$ and μ for the transfer map

$$\mu = T\delta: BG_+ \wedge BG_+ \longrightarrow BG_+.$$

Let p_G be the unique map from G to the trivial group, and write ε for Bp_{G_+} . Let β be the composite $\varepsilon \circ \mu$. In the following, D will denote the dual in the $K(h)$ -local category.

Theorem 7.6 ([Str00, 8.7, 3.11, 8.2, 8.5]). *In the $K(h)$ -local category, β is an inner product on BG_+ . Let*

$$\eta: \mathbb{S}^0 = D\mathbb{S}^0 \longrightarrow DBG_+ \xleftarrow{\cong} BG_+$$

be the composite of $D\varepsilon$ with $(\beta^\sharp)^{-1}$. Then $(BG_+, \mu, \eta, \psi, \varepsilon)$ is a Frobenius object in the $K(h)$ -local category.

From now on, let E be an even periodic $K(\mathbf{h})$ -local spectrum. Then unreduced E -cohomology of the space BG is the same as E -cohomology of the spectrum BG_+ , and we write $E^0(BG)$ for both. Let \mathfrak{m} be the composite

$$\mathfrak{m}: E^0(BG) \otimes E^0(BG) \longrightarrow E^0(BG \times BG) \longrightarrow E^0(BG)$$

of the Künneth map with ψ^* . Then $(\mathfrak{m}, \varepsilon^*)$ is the standard ring structure on $E^0(BG)$, and

$$\mathfrak{b}_G := \eta^* \circ \mathfrak{m}$$

defines a symmetric bilinear form on $E^0(BG)$. Assume that $E^0(BG)$ has finite rank over E^0 . In this case, the Künneth map becomes an isomorphism over \mathbb{Q} , and the map

$$\mu^* = \text{ind} |_{\delta}$$

defines a comultiplication on $\mathbb{Q} \otimes E^0(BG)$,

$$\mu^*: E^0(BG) \longrightarrow E^0(BG \times BG) \cong_{\mathbb{Q}} E^0(BG) \otimes E^0(BG).$$

Corollary 7.7. *If $E^0(BG)$ has finite rank over E^0 , the maps \mathfrak{m} , ε^* , μ^* and η^* make $\mathbb{Q} \otimes E^0(BG)$ into a Frobenius object in the category of $\mathbb{Q} \otimes E^0$ -modules.*

Note that \mathfrak{b}_G is defined integrally, but that there it might not satisfy the non-degeneracy condition (b) of Lemma 7.5. Note also that the augmentation map η^* is the same as the inner product with 1 :

$$\mathfrak{b}_G(\chi, 1) = \eta^*(\chi \cdot 1) = \eta^*(\chi).$$

The proof of Frobenius reciprocity [Str00, p.25] goes through (integrally) in our situation:

Proposition 7.8. *Let $i: H \rightarrow G$ be an inclusion of finite groups. Then we have*

$$\mathfrak{b}_G(\text{ind} |_{\mathbf{H}}^G \chi, \xi) = \mathfrak{b}_H(\chi, \text{res} |_{\mathbf{H}}^G \xi).$$

PROOF : Let $\xi = 1$. We have

$$\begin{aligned} \eta_G^*(\text{ind} |_{\mathbf{H}}^G \chi) &= \eta_G^*(\mathbf{T}i)^*(\chi) \\ &= \eta_G^* \circ (\beta_G^\sharp)^* \circ (D \text{Bi}_+)^* \circ ((\beta_H^\sharp)^{-1})^*(\chi) \\ &= (D\varepsilon_G)^* \circ ((\beta_G^\sharp)^{-1})^* \circ (\beta_G^\sharp)^* \circ (D \text{Bi}_+)^* \circ ((\beta_H^\sharp)^{-1})^*(\chi) \\ &= (D\varepsilon_H)^* \circ ((\beta_H^\sharp)^{-1})^*(\chi) \\ &= \eta_H^*(\chi), \end{aligned}$$

where the first equation is the definition of $\text{ind} |_{\mathbf{H}}^G$, the second equation is [Str00, 8.5], the third and the last equation follow from the definition of η , and the fourth equation follows from $p_G \circ i = p_H$ and the definition of ε . Let now ξ be arbitrary. Let $j: H \rightarrow H \times G$ denote the diagonal inclusion. Note that

$$\text{res} |_j = \text{res} |_{\delta_H} \circ (\text{id} \times \text{res} |_{\mathbf{H}}^G).$$

The proof of [Str00, 8.5] implies

$$\text{ind} |_{\mathbf{H}}^G \circ \text{res} |_j = \text{res} |_{\delta_G} \circ (\text{ind} |_{\mathbf{H}}^G \times \text{id}).$$

Combining these three equations, we obtain

$$\begin{aligned}
b_H(\chi, \text{res}|_H^G \xi) &= (\eta_H^* \circ \text{res}|_{\delta_H})(\chi, \text{res}|_H^G \xi) \\
&= \eta_G^* \circ \text{ind}|_H^G \circ \text{res}|_{\delta_H}(\chi, \text{res}|_H^G \xi) \\
&= \eta_G^* \circ \text{ind}|_H^G \circ \text{res}|_j(\chi, \xi) \\
&= \eta_G^* \circ \text{res}|_{\delta_G}(\text{ind}|_H^G \chi, \xi) \\
&= b_G(\text{ind}|_H^G \chi, \xi),
\end{aligned}$$

where the first and the last equation are the definitions of b_H and b_G . \square

7.3. Integrality theorem. The goal of this section is to prove the following proposition:

Proposition 7.9. *Over \mathbb{Q} the augmentation map $\eta^*: E^0(BG) \rightarrow E^0$ is*

$$(\eta^* \otimes \mathbb{Q})(\chi) = \frac{1}{|G|} \sum_{\alpha} \chi(\alpha).$$

Corollary 7.10. *If E^0 is torsion free, the right hand side defines a map*

$$E^0(BG) \longrightarrow E^0.$$

Corollary 7.11. *The orbifold genus*

$$\phi_{\text{orb}}(M \bowtie G) = \frac{1}{|G|} \sum_{\alpha} (\phi_G(M))(\alpha)$$

of Definition 1.1 takes values in E_h^0 .

Corollary 7.12. *The symmetric powers*

$$\sigma_n(\chi) := \frac{1}{n!} \sum_{\alpha} (\Delta_n^* P_n(\chi))(\alpha)$$

of Definition 6.11 take values in $E^0(X)$.

PROOF OF PROPOSITION 7.9: Let \mathbf{a} denote the map

$$\mathbf{a}: \chi \longmapsto \frac{1}{|G|} \sum_{\alpha} \chi(\alpha).$$

We need to show that \mathbf{a} is a unit of $\mathbb{Q} \otimes \mu^*$. Since units of co-multiplications are uniquely determined, this implies that \mathbf{a} is equal to $\eta^* \otimes \mathbb{Q}$. We first compute $\mu^* = \text{ind}|_{\delta}$ in terms of Hopkins-Kuhn-Ravenel characters. Let

$$(\alpha, \beta) := ((\mathbf{a}_1, \mathbf{b}_1), \dots, (\mathbf{a}_h, \mathbf{b}_h))$$

be an h -tuple of commuting elements of p -power order in $G \times G$. Then by Theorem 6.7

$$(\text{ind}|_{\delta}(\chi))(\alpha, \beta) = \frac{1}{|G|} \sum_{\substack{(s,t) \\ s^{-1}\alpha s = t^{-1}\beta t}} \chi(s^{-1}\alpha s).$$

Thus, counting the pairs (s, t) and taking into account that $\chi(s^{-1}\alpha s) = \chi(\alpha)$, we have

$$(\mu^*(\chi))(\alpha, \beta) = \frac{1}{|G|} \sum_{s \in G} \sum_{\substack{t \in G \\ s^{-1}\alpha s = t^{-1}\beta t}} \chi(\alpha) = \begin{cases} |C_\alpha| \cdot \chi(\alpha) & \alpha \sim_G \beta \\ 0 & \text{else.} \end{cases}$$

We are now ready to prove that \mathbf{a} is a unit of $\mu^* \otimes \mathbb{Q}$, i.e. that the equality

$$(\text{id}_{E_h^0(BG)} \otimes \mathbf{a}) \circ \mu^* = \text{id}_{E_h^0(BG)}$$

holds over \mathbb{Q} . By Theorem 6.6 it suffices to show that both sides define the same class function. Write

$$\xi := (\text{id} \otimes \mathbf{a}) \circ \mu^*(\chi).$$

We have

$$\begin{aligned} \xi(\alpha) &= \frac{1}{|G|} \sum_{\beta} (\mu^*(\chi))(\alpha, \beta) \\ &= \frac{1}{|G|} \sum_{\beta \in [\alpha]_G} |C_\alpha| \cdot \chi(\alpha) \\ &= \chi(\alpha). \end{aligned}$$

□

As a further corollary of Proposition 7.9 we obtain the formula for the Strickland inner product mentioned in the introduction:

Corollary 7.13. *Let E be a cohomology theory with Hopkins-Kuhn-Ravenel theory, and assume that E^0 is torsion free. Then the Strickland inner product on $E^0(BG)$ is described by the formula*

$$\mathbf{b}_G(\chi, \xi) = \frac{1}{|G|} \sum_{\alpha} \chi(\alpha) \xi(\alpha).$$

PROOF : We have

$$\mathbf{b}_G(\chi, \xi) = (\eta^* \circ \mathbf{m})(\chi, \xi) = \eta^*(\chi \cdot \xi) = \frac{1}{|G|} \sum_{\alpha} \chi(\alpha) \xi(\alpha),$$

where the first equation is the definition of \mathbf{b}_G , the second is the fact that \mathbf{m} is the standard multiplication on $E^0(BG)$, and the last equation follows from Proposition 7.9, since E^0 is torsion free. □

7.4. Generalized orbifold genera. We are now ready to give our most general definition of orbifold genus. Recall that Definition 1.1 requires an even periodic cohomology theory E with level h Hopkins-Kuhn-Ravenel theory, and that any such E is $K(h)$ -local. Proposition 7.9 motivates the following definition:

Definition 7.14. Let E be an even periodic $K(h)$ -local ring spectrum, and let $\phi: MU \rightarrow E$ be a map of ring spectra. Let G be a finite group, and let ϕ_G be the Borel equivariant genus associated to ϕ as in Definition 1.1. We define the orbifold genus ϕ_{orb} of stably almost complex G manifolds as the composition

$$\phi_{\text{orb}} := \eta^* \circ \phi_G : \mathcal{N}_*^{\text{u}, G} \longrightarrow E_*,$$

where η is the map of Theorem 7.6.

Instead of MU we could have used any of the classical Thom spectra $MSpin$, MO , MSp , $MU\langle n \rangle$, $MO\langle n \rangle$ etc. In the case $E = E_h$, Definition 7.14 specializes by Proposition 7.9 to the Ando-French Definition 1.1.

7.5. Generalized symmetric powers. Recall the definitions of symmetric powers in K -theory (Example 4.9) and in E -cohomology, where E is an H_∞ -spectrum with Hopkins-Kuhn-Ravenel theory and a Künneth isomorphism for the symmetric groups (Definition 6.11). Proposition 7.9 motivates the following generalization of these definitions:

Definition 7.15. Let E be an even periodic $K(h)$ -local H_∞ -ring spectrum. Let X be a space with basepoint or a spectrum. We define the n^{th} symmetric power in $E(X)$ by

$$\sigma_n := (\eta_{\Sigma_n} \wedge \text{id}_X)^* \circ \Delta_n^* \circ P_n$$

and the total symmetric power by

$$S_t: E(X) \longrightarrow \bigoplus_{n \geq 0} E(E\Sigma_{n+} \wedge_{\Sigma_n} X^n) t^n \longrightarrow \bigoplus_{n \geq 0} E(B\Sigma_{n+} \wedge X) t^n \longrightarrow E(X)[[t]]$$

$$S_t(x) = \sum_{n=0}^{\infty} \sigma_n t^n(x),$$

where the first map is the total power operation, and on the n^{th} summand, the second map is pullback along the diagonal of X^n , while the third map is pullback along $\eta_{\Sigma_n} \wedge \text{id}_X$.

In the situation of Definition 6.11, Theorem 7.3 implies that E is $K(h)$ -local, and the two definitions agree by Proposition 7.9. Note that Definition 7.15 does not require a Künneth condition like the one in Definition 6.11. We are now going to show that S_t is exponential. Recall from Section 4.2 that

$$\bigoplus_{n \geq 0} E^0(E\Sigma_{n+} \wedge_{\Sigma_n} X^n) t^n$$

is a ring, where multiplication is defined using the transfer maps $\text{ind}_{\Sigma_m \times \Sigma_n}^{\Sigma_{n+m}}$,

$$E_{\Sigma_n}(X^n) \otimes E_{\Sigma_m}(X^m) \longrightarrow E_{\Sigma_n \times \Sigma_m}(X^n \times X^m) \longrightarrow E_{\Sigma_{n+m}}(X^{n+m}).$$

Lemma 7.16. *The map*

$$\left(\sum_{n \geq 1} (\eta_{\Sigma_n} \wedge \text{id}_X)^* \right) \circ \left(\sum_{n \geq 1} \Delta_n^* \right): \bigoplus_{n \geq 0} E^0(E\Sigma_{n+} \wedge_{\Sigma_n} X^n) t^n \longrightarrow E^0(X)[[t]]$$

is a map of rings.

Corollary 7.17. *The total symmetric power S_t takes sums into products.*

PROOF : Total power operations take sums into products (cf. Proposition 4.5 (c)), and S_t is defined as a total power operation followed by the ring map of the lemma. \square

PROOF OF LEMMA 7.16: Note first that the target of $\sum_{n \geq 0} \Delta_n^*$,

$$\bigoplus_{n \geq 0} E^0(B\Sigma_{n+} \wedge_{\Sigma_n} X) t^n,$$

also carries a ring structure: the multiplication is defined by

$$E_{\Sigma_n}(X) \otimes E_{\Sigma_m}(X) \longrightarrow E_{\Sigma_n \times \Sigma_m}(X \times X) \xrightarrow{\Delta_2^*} E_{\Sigma_n \times \Sigma_m}(X) \longrightarrow E_{\Sigma_{n+m}}(X),$$

where E_G denotes Borel equivariant E-cohomology and the last map is again $\text{ind} |_{\Sigma_n \times \Sigma_m}^{\Sigma_{n+m}}$. We have

$$\Delta_{n+m} = (\Delta_n \times \Delta_m) \circ \Delta_2$$

as maps of Σ_{n+m} -spaces, and $\text{ind} |_H^G$ is natural in maps of G -spaces. Therefore the map

$$\sum_{n \geq 1} \Delta_n^*$$

is a ring map. It remains to show that

$$\sum_{n \geq 1} (\eta_{\Sigma_n} \wedge \text{id}_X)^*$$

is a map of rings. Recall that

$$\varepsilon_G: BG_+ \longrightarrow \mathbb{S}^0$$

is $B(-)_+$ applied to the unique map from G to the trivial group, and that $\eta_G = D\varepsilon_G$. Thus

$$\varepsilon_{G \times H} = \varepsilon_G \wedge \varepsilon_H \quad \text{and} \quad \eta_{G \times H} = \eta_G \wedge \eta_H.$$

Together with Frobenius reciprocity, this implies

$$\begin{aligned} \eta_{\Sigma_{n+m}}^* \circ \text{ind} |_{\Sigma_n \times \Sigma_m}^{\Sigma_{n+m}} &= (\tau_{\Sigma_n \times \Sigma_m}^{\Sigma_{n+m}} \circ \eta_{\Sigma_{n+m}})^* \\ &= \eta_{\Sigma_n \times \Sigma_m}^* \\ &= (\eta_{\Sigma_n} \wedge \eta_{\Sigma_m})^*. \end{aligned}$$

This proves the lemma if X is a point. Together,

$$\begin{aligned} (\eta_{\Sigma_{n+m}} \wedge \text{id}_X)^* \circ (\tau_{\Sigma_n \times \Sigma_m}^{\Sigma_{n+m}} \wedge \Delta_2)^* &= ((\eta_{\Sigma_n} \wedge \eta_{\Sigma_m}) \wedge \Delta_2)^* \\ &= (\text{id}_{\mathbb{S}^0} \wedge \Delta_2)^* \circ (\eta_{\Sigma_n} \wedge \eta_{\Sigma_m} \wedge \text{id}_{X \wedge X})^*. \end{aligned}$$

□

8. THE ORBIFOLD GENUS ϕ_{orb} AS ORBIFOLD INVARIANT

Let G and H be finite groups acting on smooth manifolds M and N respectively. In this section we recall the notion of tangentially almost complex structure and prove the following theorem:

Theorem 8.1. *If the orbifold quotients $M//G$ and $N//H$ are isomorphic as orbifolds with almost complex structure, then*

$$\phi_{\text{orb}}(M \bowtie G) = \phi_{\text{orb}}(N \bowtie H).$$

Note however that our definition of ϕ_{orb} only makes sense for orbifolds which can be represented as a global quotient $M//G$ by a finite group G .

Remark 8.2. For Borisov and Libgober's definition of the orbifold elliptic genus the analogous statement is a consequence of the McKay correspondence, proved in [BL02].

We use the following facts about orbifolds: an isomorphism of orbifolds

$$\mathbf{M} // \mathbf{G} \cong \mathbf{N} // \mathbf{H}$$

induces an isomorphism of (real or complex) equivariant \mathbf{K} -groups

$$\mathbf{K}_{\mathbf{G}}(\mathbf{M}) \cong \mathbf{K}_{\mathbf{H}}(\mathbf{N})$$

and a homotopy equivalence of Borel constructions

$$\mathbf{E}\mathbf{G} \times_{\mathbf{G}} \mathbf{M} \simeq \mathbf{E}\mathbf{H} \times_{\mathbf{H}} \mathbf{N},$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathbf{K}_{\mathbf{G}}(\mathbf{M}) & \longrightarrow & \mathbf{K}(\mathbf{E}\mathbf{G} \times_{\mathbf{G}} \mathbf{M}) \\ \parallel & & \parallel \\ \mathbf{K}_{\mathbf{H}}(\mathbf{N}) & \longrightarrow & \mathbf{K}(\mathbf{E}\mathbf{H} \times_{\mathbf{H}} \mathbf{N}). \end{array}$$

Here the horizontal arrows are the completion maps, and we will use the notation

$$\text{Borel}: \mathbf{K}_{\text{orb}}(\mathbf{M} // \mathbf{G}) \longrightarrow \mathbf{K}(\text{Borel}(\mathbf{M} // \mathbf{G}))$$

if we want to emphasize its independence of the representation of the orbifold. For more background on orbifolds, we refer the reader to [Moe02].

8.1. Tangentially almost complex structures. Recall that a (stably) almost complex structure on a \mathbf{G} -manifold \mathbf{M} is a choice of lift $-\lceil \tau \rceil_{\mathbf{K}} \in \tilde{\mathbf{K}}_{\mathbf{G}}(\mathbf{M})$ of the stable normal bundle $-\lceil \tau \rceil \in \widetilde{\mathbf{K}\mathbf{O}}_{\mathbf{G}}(\mathbf{M})$. The tangent vector bundle is a well-defined orbifold notion [Sat57], but

$$\tilde{\mathbf{K}}_{\mathbf{G}}(\mathbf{M}) = \text{coker}(\mathbf{K}_{\mathbf{G}}(\text{pt}) \longrightarrow \mathbf{K}_{\mathbf{G}}(\mathbf{M}))$$

is not, since there is no fixed group \mathbf{G} . We can however define

$$\check{\mathbf{K}}_{\text{orb}} \underline{\mathbf{X}} := \text{coker}(\mathbf{K}_{\text{orb}}(\text{pt}) \longrightarrow \mathbf{K}_{\text{orb}}(\underline{\mathbf{X}})),$$

for an arbitrary orbifold $\underline{\mathbf{X}}$, and similarly $\check{\mathbf{K}\mathbf{O}}_{\text{orb}}$.

Definition 8.3 (compare [May96, XXVIII.3.1]). A tangentially almost complex structure on an orbifold $\underline{\mathbf{X}}$ is a choice of lift $\lceil \tau \rceil_{\check{\mathbf{K}}} \in \check{\mathbf{K}}_{\text{orb}}(\underline{\mathbf{X}})$ of $\lceil \tau \rceil \in \check{\mathbf{K}\mathbf{O}}_{\text{orb}}(\underline{\mathbf{X}})$.

The reduced completion map

$$\check{\mathbf{K}}_{\text{orb}}(\mathbf{M} // \mathbf{G}) \longrightarrow \check{\mathbf{K}}(\text{Borel}(\mathbf{M} // \mathbf{G}))$$

sends a tangentially almost complex structure on $\mathbf{M} // \mathbf{G}$ to a lift

$$-\lceil \text{Borel}(\tau) \rceil_{\check{\mathbf{K}}} \in \check{\mathbf{K}}(\text{Borel}(\mathbf{M} // \mathbf{G}))$$

in such a way that the Borel-equivariant Thom isomorphism

$$\mathbf{E}^0(\mathbf{E}\mathbf{G} \times_{\mathbf{G}} \mathbf{M}) \longrightarrow \mathbf{E}^{-d}(\mathbf{E}\mathbf{G} \times_{\mathbf{G}} \mathbf{M}^{-\tau})$$

defined by $-\lceil \tau \rceil_{\check{\mathbf{K}}}$ agrees with the non-equivariant Thom isomorphism

$$\mathbf{E}^0(\text{Borel}(\mathbf{M} // \mathbf{G})) \longrightarrow \mathbf{E}^{-d}((\text{Borel}(\mathbf{M} // \mathbf{G}))^{-\text{Borel}(\tau)})$$

defined by $-\lceil \text{Borel}(\tau) \rceil_{\check{\mathbf{K}}}$ (compare Section 3.2). In particular, this Thom isomorphism is independent of the representation of the orbifold.

8.2. **The genus ϕ_{orb} as orbifold invariant.** Recall from Section 7.4 that

$$\phi_{\text{orb}}(\mathbf{M} \bowtie \mathbf{G}) \in E_d$$

is the image of one under the composite

$$(7) \quad E^0(EG \times_G \mathbf{M}) \longrightarrow E^{-d}(EG \times_G \mathbf{M}^{-\tau}) \xrightarrow{\text{PT}} E^{-d}BG \xrightarrow{\eta^*} E^{-d}\mathbb{S}^0,$$

where the first map is the Thom isomorphism defined by $-[\tau]_{\tilde{K}}$, the second map is the Pontrjagin-Thom collapse and the third map is pullback along the map η of Section 7.2. We have just seen that the Thom isomorphism is independent of the representation of $\mathbf{M} // \mathbf{G}$, as long as we fix a tangentially almost complex structure. The second and third map in (7) clearly depend on the representation of the orbifold, because BG does. However, we will show that the composition

$$(8) \quad \text{Borel}(\text{PT}) \circ \eta : \mathbb{S}^0 \longrightarrow EG \times_G \mathbf{M}^{-\tau}$$

in the $K(\mathbf{h})$ -local category is independent of the representation. More precisely, we will prove the following theorem:

Theorem 8.4. *The map (8) is the Spanier-Whitehead dual in the $K(\mathbf{h})$ -local category of the map*

$$(EG \times_G \mathbf{M})_+ \longrightarrow \mathbb{S}^0,$$

sending $EG \times_G \mathbf{M}$ to the non-basepoint of \mathbb{S}^0 .

As a corollary, we obtain Theorem 8.1.

PROOF OF THEOREM 8.1: The map of Theorem 8.4 is independent of the representation of $\mathbf{M} // \mathbf{G}$. Therefore, so are the maps in (8) and (7). \square

The remainder of this section is devoted to the proof of Theorem 8.4.

8.3. **Borel construction and duality.** We retain the notation of Section 7.1 and write

$$D_G(-) := F_G(-, \mathbb{S}^0)$$

for the G -equivariant dual (cf. [May96, XVI.7]) and

$$D(-) := F_{S_{K(\mathbf{h})}}(-, \mathbb{S}^0)$$

for the dual in the $K(\mathbf{h})$ -local category.

Theorem 8.5. *Let G be a finite group and let Y be a finite G -CW spectrum. There is an isomorphism in the $K(\mathbf{h})$ -local category*

$$(9) \quad EG \times_G (D_G(Y)) \longrightarrow D(EG \times_G Y),$$

which is natural in Y .

In order to construct the map in (9), we construct its adjoint

$$(10) \quad (EG \times_G (D_G(Y))) \wedge (EG \times_G Y) \longrightarrow \mathbb{S}^0.$$

We construct (10) as a map in the (non-localized) stable homotopy category, but the adjunction that yields (9) is in the $K(\mathbf{h})$ -local category. Recall that for a finite group G and G -spectra X and Y there is an isomorphism of functors

$$(EG \times_G X) \wedge (EG \times_G Y) \cong (EG \times EG) \times_{G \times G} (X \wedge Y).$$

The map (10) is defined as the composite of several maps: the first is transfer along the diagonal δ of G

$$\mathrm{T}\delta: (EG \times EG) \ltimes_{G \times G} (D_G(Y) \wedge Y) \longrightarrow (EG \times EG) \ltimes_G (D_G(Y) \wedge Y).$$

The second map is

$$(EG \times EG) \ltimes_G -$$

applied to the canonical G -map

$$(11) \quad \beta_G: D_G Y \wedge Y \longrightarrow \mathbb{S}^0$$

(cf. Theorem 7.1). Its target is (the suspension spectrum of)

$$(EG \times EG)_+ \wedge_G \mathbb{S}^0 \simeq BG_+.$$

The last map is

$$(Bp_G)_+ : BG_+ \longrightarrow \mathbb{S}^0,$$

where p_G denotes the unique map from G to the trivial group. This completes the construction of the map (10).

Remark 8.6. In the case $Y = \mathbb{S}^0$, the construction of (10) specializes to the definition of Strickland's inner product [Str00, 8.2]

$$\beta: BG_+ \wedge BG_+ \longrightarrow \mathbb{S}^0.$$

Corollary 8.7. *Theorem 8.5 is true for $Y = \mathbb{S}^0$.*

PROOF : This is the fact that the Strickland inner product is non-degenerate [Str00, 8.3]. \square

The second easiest special case of Theorem 8.5 is the case that Y is a different zero sphere.

Proposition 8.8. *For $Y = G/H_+$ the map (10) is the Strickland inner product on*

$$(EG \times_G G/H)_+,$$

and the theorem holds for $Y = G/H_+$.

PROOF : Recall from [May96, p.176] that for finite groups $H \subseteq G$ a G -equivariant (strong) dual of G/H_+ is G/H_+ , with the map β_G in (11) given by the composite (of space level maps)

$$(12) \quad (G/H \times G/H)_+ \longrightarrow G/H_+ \longrightarrow \mathbb{S}^0,$$

where the first map is the G -equivariant Pontrjagin-Thom collapse along the diagonal inclusion (that is, it is the identity on the diagonal and everything else gets mapped to the basepoint) and the second map is p_+ , where p is the unique (G -equivariant) map

$$p: G/H \longrightarrow \mathrm{pt}.$$

Following Strickland, we write $B\mathcal{G}$ for the Borel construction of the finite groupoid \mathcal{G} defined by the action of G on G/H , remembering that

$$(B\mathcal{G})_+ = EG_+ \wedge_G G/H_+.$$

Strickland defines the inner product on (the suspension spectrum of) $(B\mathcal{G})_+$ as the composite

$$\beta: (B\mathcal{G} \times B\mathcal{G})_+ \xrightarrow{\mathrm{T}\delta} (B\mathcal{G})_+ \longrightarrow \mathbb{S}^0,$$

where

$$\delta = \delta_{\mathcal{G}}: \mathcal{G} \longrightarrow \mathcal{G} \times \mathcal{G}$$

is the diagonal inclusion of groupoids, and the second map is the Borel construction of the unique map of groupoids

$$p_{\mathcal{G}}: \mathcal{G} \longrightarrow 1.$$

Note that $\delta_{\mathcal{G}}$ factors as

$$\delta_{\mathcal{G}}: (G/H \bowtie G) \xrightarrow{i} (G/H \times G/H) \bowtie G \hookrightarrow (G/H \times G/H) \bowtie G \times G,$$

where i is an inclusion of finite G -sets (namely the diagonal inclusion mentioned above), and the second map is the diagonal inclusion of groups δ_G , whose transfer $T\delta_G$ is the first map in the construction of (10). We need to identify Ti . However, Bi is a particularly simple example of covering with finite fibers, namely the inclusion of some path components. A look at the construction of Ti in [Ada78, 4.1.1] shows that Ti is given by the (space level) map

$$(EG \times_G (G/H \times G/H))_+ \longrightarrow (EG \times_G G/H)_+$$

that is the identity on $\text{im}(Bi)$ and maps everything else to the basepoint. This is exactly $EG_+ \wedge_G -$ applied to the Pontrjagin-Thom collapse in (12). The map $p_{\mathcal{G}}$ factors as

$$p_{\mathcal{G}}: (G/H \bowtie G) \xrightarrow{p} (\text{pt} \bowtie G) \xrightarrow{p_G} (\text{pt} \bowtie 1),$$

where p is as in (12). Together this proves the claim that (10) is the Strickland inner product:

$$\beta = (Bp_{\mathcal{G}})_+ \circ T\delta_{\mathcal{G}} = (Bp_G)_+ \circ (Bp)_+ \circ Ti \circ T\delta_G = (Bp_G)_+ \circ (EG_+ \wedge_G \beta_G) \circ T\delta_G.$$

As above, non-degeneracy of the Strickland product in the $K(\mathfrak{h})$ -local category implies that (9) is an isomorphism for $Y = G/H_+$. \square

Before we proceed to higher dimensional spheres, we recall that in any closed symmetric monoidal category we have an isomorphism

$$(13) \quad D(X) \wedge D(Y) \xrightarrow{\cong} D(X \wedge Y),$$

which identifies the evaluation map $\beta_{X \wedge Y}$ with

$$(\beta_X \wedge \beta_Y) \circ (\text{id}_{DX} \wedge \tau \wedge \text{id}_Y),$$

where τ switches DY and X .

Lemma 8.9. *Theorem 8.5 holds for spheres*

$$Y = G/H_+ \wedge \mathbb{S}^n.$$

PROOF : The sphere \mathbb{S}^n is strongly dualizable in \mathcal{S} with dual \mathbb{S}^{-n} , and both functors $\mathcal{S} \rightarrow \mathcal{S}_G$ and $\mathcal{S} \rightarrow \mathcal{S}_{K(\mathfrak{h})}$ preserve the data of strong dualizability in Theorem 7.1. By (13) we have

$$D_G(G/H_+ \wedge \mathbb{S}^n) \cong D_G(G/H_+) \wedge \mathbb{S}^{-n}.$$

Since \mathbb{S}^n and \mathbb{S}^{-n} have trivial G -action, we have

$$EG \times_G (G/H_+ \wedge \mathbb{S}^{\pm n}) = (EG_+ \wedge_G G/H_+) \wedge \mathbb{S}^{\pm n},$$

and under this identification the map (10) becomes

$$\beta_{EG_+ \wedge_G (G/H_+)} \wedge \beta_{\mathbb{S}^n} : EG_+ \wedge_G (D_G(G/H_+)) \wedge EG_+ \wedge_G (G/H_+) \wedge \mathbb{S}^n \wedge \mathbb{S}^{-n} \longrightarrow \mathbb{S}^0.$$

Here $\beta_{EG_+ \wedge_G (G/H_+)}$ is the map of the theorem for G/H_+ , and by Proposition 8.8, it is the evaluation map of a strong duality in $\mathcal{S}_{K(\mathfrak{h})}$. The map $\beta_{\mathbb{S}^n}$ is already an evaluation of a strong duality in \mathcal{S} and thus also in $\mathcal{S}_{K(\mathfrak{h})}$. We apply (13) again, this time in the $K(\mathfrak{h})$ -local

category, to complete the proof. \square

PROOF OF THEOREM 8.5: We prove the theorem by induction over the cells. All our categories have compatible triangulated and closed symmetric monoidal structures. In particular, duals commute with direct sums and take triangles (in the opposite category) into triangles. The Borel construction also preserves the triangulated structure. Since both sides of (9) preserve finite sums, Lemma 8.9 implies that the statement is true for finite bouquets of spheres. Both sides of (9) preserve triangles, thus if the theorem is true for two objects in an exact triangle, it is also true for the third. \square

PROOF OF THEOREM 8.4: The map of Theorem 8.4 is the Borel construction

$$\text{Borel}(\mathfrak{p}_{M//G})_+$$

of the unique map of orbifolds from $M//G$ to a point. This map factors as

$$\mathfrak{p}_{M//G} : M//G \xrightarrow{\pi} \text{pt} // G \xrightarrow{\mathfrak{p}_G} \text{pt},$$

where π is the unique G -map from M to a point, and \mathfrak{p}_G is the unique map from G to the trivial group. Recall from Theorem 7.6 that η is defined as

$$\eta : \mathbb{S}^0 \xrightarrow{\cong} D\mathbb{S}^0 \longrightarrow D(\text{B}G_+) \xleftarrow{\cong} \text{B}G_+,$$

where the second map is $D((\text{B}\mathfrak{p}_G)_+)$, and the last map is the adjoint β^\sharp of the Strickland inner product on $\text{B}G_+$.

By [May96, XVI.8.1], the Thom spectrum $M^{-\tau}$ is a G -equivariant (strong) dual of M_+ , and the G -equivariant Pontrjagin-Thom collapse

$$\text{PT} : \mathbb{S}^0 \mathfrak{S} G \longrightarrow M^{-\tau} \mathfrak{S} G$$

is the dual $D_G(\pi_+)$. Under the isomorphism of Theorem 8.5 (vertical arrows in the diagram below), the twisted half smash product $EG \ltimes_G (\text{PT})$ (top row) becomes

$$\begin{array}{ccccc} EG \ltimes_G (D_G M_+) & \longleftarrow & EG_+ \wedge_G (D_G \mathbb{S}^0) & \xleftarrow{\cong} & EG_+ \wedge_G \mathbb{S}^0 \\ \cong \downarrow & & \downarrow \cong & & \\ D(EG_+ \wedge_G M_+) & \longleftarrow & D(EG_+ \wedge_G \mathbb{S}^0), & & \end{array}$$

where the bottom arrow is $D(EG_+ \wedge_G \pi_+)$. By Remark 8.6, the composite of the two rightmost arrows is β^\sharp . When precomposing with η , β^\sharp and its inverse cancel out, and we obtain

$$EG \ltimes_G (\text{PT}) \circ \eta = D((EG \times_G \pi)_+) \circ D((\text{B}\mathfrak{p}_G)_+) = D(\text{Borel}(\mathfrak{p}_{M//G})_+),$$

which completes the proof. \square

9. THE DMVV FORMULA

9.1. Conjugacy classes of h -tuples of commuting elements of Σ_l . Just as the conjugacy classes of elements of Σ_l are in one to one correspondence with partitions

$$\sum_{31} a_n n = l$$

(i.e. the shape of the Young tableau), one also describes conjugacy classes of \mathbf{h} -tuples of commuting elements in terms of the corresponding orbit decomposition of the set

$$\underline{l} := \{1, \dots, l\}.$$

More precisely, such an \mathbf{h} -tuple (g_1, \dots, g_h) defines an action of \mathbb{Z}^h on \underline{l} , and \underline{l} decomposes into orbits of that action. Two such \mathbf{h} -tuples are conjugate by a permutation g of \underline{l} if and only if their orbit decompositions are isomorphic (and an isomorphism is given by g).

Orbits are finite transitive \mathbb{Z}^h -sets, and every finite transitive \mathbb{Z}^h -set T turns up as a possible orbit for $l \geq |T|$, where $|T|$ is the number of elements in T .

Let $\mathcal{T} = \{T\}$ contain one representative for each isomorphism class of finite transitive \mathbb{Z}^h -sets. The above discussion summarizes as follows. The conjugacy classes of \mathbf{h} -tuples of commuting elements in Σ_l are classified by expressions

$$\sum_{T \in \mathcal{T}} a_T T \text{ s.t. } \sum_{T \in \mathcal{T}} a_T |T| = l,$$

where for given (g_1, \dots, g_h) the expression $\sum_{T \in \mathcal{T}} a_T T$ counts the number a_T of times each isomorphism class of finite transitive \mathbb{Z}^h -set T occurs in the decomposition of \underline{l} into orbits of the subgroup $\langle g_1, \dots, g_h \rangle$ generated by the g_i . If the conjugacy class $[g_1, \dots, g_h]$ corresponds to $\sum_{T \in \mathcal{T}} a_T T$, then the centralizer of α in G can be described as follows

$$(14) \quad C_{(g_1, \dots, g_h)} \cong \prod_{T \in \mathcal{T}} \text{Aut}_{\mathbb{Z}^h}(T)^{a_T} \rtimes \Sigma_{a_T},$$

where Σ_{a_T} permutes the a_T orbits isomorphic to T and $\text{Aut}_{\mathbb{Z}^h}(T)$ acts on each of them individually. Now T is a transitive \mathbb{Z}^h -set, and \mathbb{Z}^h is abelian. This means that for any $x, y \in T$ there is an element $z \in \mathbb{Z}^h$ such that $zx = y$, and multiplication with z is the unique automorphism of T mapping x to y . Thus the number of elements in $\text{Aut}_{\mathbb{Z}^h}(T)$ is

$$|\text{Aut}_{\mathbb{Z}^h}(T)| = |T|.$$

Since the conjugacy class of (g_1, \dots, g_h) in $\text{Hom}(\mathbb{Z}^h, \Sigma_l)$ is the orbit of (g_1, \dots, g_h) under the action of Σ_l by conjugation, we have

$$[g_1, \dots, g_h]_{\Sigma_l} \cong_{\Sigma_l} \Sigma_l / C_{(g_1, \dots, g_h)}.$$

Therefore, its number of elements is by (14)

$$|[g_1, \dots, g_h]_{\Sigma_l}| = \frac{l!}{\prod_{T \in \mathcal{T}} |T|^{a_T} a_T}.$$

Assume now that we are only interested in \mathbf{h} -tuples of commuting elements of p -power order. Then the same discussion goes through, but we need to replace \mathcal{T} by the set \mathcal{T}_p containing one representative for each isomorphism class of finite transitive \mathbb{Z}_p^h -set. Note that elements of \mathcal{T}_p have p -power cardinalities, since each of them can be identified with a quotient of $(\mathbb{Z}/p^j\mathbb{Z})^h$ for some sufficiently large j .

9.2. The DMVV formula for ϕ_{orb} . We start by proving a formula for the total symmetric power. Let S_t be as in Definition 6.11 and T_{p^k} as in Definition 6.10.

Proposition 9.1. *We have*

$$S_t(x) = \exp \left[\sum_{k \geq 0} T_{p^k}(x) t^{p^k} \right].$$

PROOF : We have

$$\exp \left[\sum_{k \geq 0} T_{p^k}(x) t^{p^k} \right] = \sum_{m \geq 0} \frac{1}{m!} \left[\sum_{k \geq 0} T_{p^k}(x) t^{p^k} \right]^m.$$

In this equation, the coefficient of t^l is

$$\sum_{\substack{l = \sum_{T \in \mathcal{I}_p} a_T \\ a_T \geq 0}} \frac{(\sum a_T)!}{\prod (a_T!)} \frac{1}{(\sum a_T)!} \prod_{T \in \mathcal{I}_p} \left(\frac{\psi_T(x)}{|T|} \right)^{a_T},$$

where $\frac{(\sum a_T)!}{\prod (a_T!)}$ counts the number of ways to partition a set of $\sum a_T$ (orbits) into subsets of size a_T (the number of times T occurs as orbit), and $\frac{1}{(\sum a_T)!}$ is $\frac{1}{m!}$. This is

$$\begin{aligned} \sum_{\sum a_T |T|=l} \prod_{T \in \mathcal{I}_p} \frac{1}{|T|^{a_T} (a_T!)} \psi_T(x)^{a_T} &= \sum_{\sum a_T |T|=l} \left(\prod_{T \in \mathcal{I}_p} \frac{1}{|T|^{a_T} (a_T!)} \right) \psi_{(\prod_{T \in \mathcal{I}_p} a_T T)}(x) \\ &= \sum_{[\alpha]} \frac{1}{|C_\alpha|} \psi_\alpha(x) \\ &= \sigma_l(x). \end{aligned}$$

□

We are now ready to prove Theorem 1.5 of the introduction.

Theorem 9.2. *Let ϕ be an H_∞ -orientation of E_h . Then*

$$\sum_{n \geq 0} \phi_{\text{orb}}(M^n // \Sigma_n) t^n = \exp \left[\sum_{k \geq 0} T_{p^k}(\phi(M)) t^{p^k} \right].$$

PROOF : We have

$$\begin{aligned} \sum_{n \geq 0} \phi_{\text{orb}}(M^n // \Sigma_n) t^n &= \sum_{n \geq 0} \eta_{\Sigma_n}^* \circ \phi_{\Sigma_n}(M^n) t^n \\ &= \sum_{n \geq 0} \eta_{\Sigma_n}^* \circ \phi_{\Sigma_n} \circ P_n^{\text{MU}}(M) t^n \\ &= \sum_{n \geq 0} \eta_{\Sigma_n}^* \circ P_n^{E_h} \circ \phi(M) t^n \\ &= S_t(\phi(M)), \end{aligned}$$

where the first equation is Definition 1.1, P_n^{MU} and $P_n^{E_h}$ are the n^{th} power operations in cobordism and Morava E-theory of a point, the third equation holds, because ϕ is an H_∞ -map, and the fourth equation is the definition of S_t . The claim now follows from Proposition 9.1. □

Note the striking similarity of the right hand side of the DMVV-formula with the formal inverse of Rezk's logarithm formula [Rez, p.4]

$$\exp \sum_{k \geq 0} T_{p^k}(-).$$

Here the T_{p^k} are as in Definition 6.10.

Example 9.3 (σ -orientation). Any elliptic spectrum E has a canonical orientation

$$\sigma_E: MU\langle 6 \rangle \longrightarrow E,$$

and it was shown in [AHS04], that in the case $E = E_2$, the map σ is an H_∞ -map.

The following result due to Ando classifies the complex genera into E_h that can be taken as input for Theorem 1.5:

Theorem 9.4 ([And95]). *The spectrum E_h is an H_∞ -spectrum. A map of ring spectra*

$$\phi: MU \longrightarrow E_h$$

is an H_∞ -map if and only if the p -series of its Euler class e_ϕ (of the universal line bundle) satisfies

$$[p]_F(e_\phi) = \prod_{\substack{v \in F(D_1) \\ [p]_F(v)=0}} (v +_F e_\phi),$$

where D_1 denotes the ring extension of E_n^0 obtained by adjoining the roots of the p -series of F , and $F(D_1)$ stands for the maximal ideal of D_1 with the group structure $x +_F y$.

9.3. Atiyah-Tall-Grothendieck type definition of Hecke operators. The left-hand side of the equation in Proposition 9.1 is defined in greater generality than its right-hand side, motivating the following definition. Let E be an even periodic, $K(h)$ -local H_∞ -spectrum. Then the total symmetric power S_t is defined on elements of $E(X)$ and takes values in

$$1 + tE(X)[[t]]$$

(cf. Definition 7.15).

Definition 9.5. In this situation we define additive operators T_n on $E(X)$ by

$$\sum_{n \geq 1} T_n t^n := \log S_t.$$

Following Grothendieck [Gro57], or the interpretation for K -theory by Atiyah and Tall [AT69], we note that

$$t \frac{d}{dt} \log S_t(x) = t \frac{\frac{d}{dt} S_t(x)}{S_t(x)}.$$

takes values in $E(X)[[t]]$. Thus the Hecke operators are operations

$$T_n: E(X) \longrightarrow \frac{1}{n} E(X).$$

We can make the connection to the Atiyah-Tall-Grothendieck definition of the Adams operations even more precise: Let

$$\Lambda_t := \frac{1}{S_{-t}}$$

denote the “total exterior power” in E-theory. This defines a λ -ring structure on $E(X)$, whose Adams operations are given by $\psi_n = nT_n$.

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